Tuesday, FEBRUARY 10, 2009

60th Annual American Mathematics Contest 12

AMC 12
CONTEST A

THE MATHEMATICAL ASSOCIATION OF AMERICA
American Mathematics Competitions

1. DO NOT OPEN THIS BOOKLET UNTIL YOUR PROCTOR GIVES THE SIGNAL TO BEGIN.
2. This is a 25-question, multiple choice test. Each question is followed by answers marked A, B, C, D and E. Only one of these is correct.
3. Mark your answer to each problem on the AMC 12 Answer Form with a #2 pencil. Check the blackened circles for accuracy and erase errors and stray marks completely. Only answers properly marked on the answer form will be graded.
4. SCORING: You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. No aids are permitted other than scratch paper, graph paper, ruler, compass, protractor, and erasers. No calculators are allowed. No problems on the test will *require* the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the test, your proctor will ask you to record certain information on the answer form. When your proctor gives the signal, begin working the problems. You will have **75 MINUTES** to complete the test.
8. When you finish the exam, *sign your name* in the space provided on the Answer Form.

Students who score 100 or above or finish in the top 5% on this AMC 12 will be invited to take the 27th annual American Invitational Mathematics Examination (AIME) on Tuesday, March 17, 2009 or Wednesday, April 1, 2009. More details about the AIME and other information are on the back page of this test booklet.

The Committee on the American Mathematics Competitions (CAMC) reserves the right to re-examine students before deciding whether to grant official status to their scores. The CAMC also reserves the right to disqualify all scores from a school if it is determined that the required security procedures were not followed.

1. Kim's flight took off from Newark at 10:34 AM and landed in Miami at 1:18 PM. Both cites are in the same time zone. If her flight took h hours and m minutes, with $0 \le m < 60$, what is $h + m$?

(A) 46 **(B)** 47 **(C)** 50 **(D)** 53 **(E)** 54

2. Which of the following is equal to $1 + \dfrac{1}{1 + \frac{1}{1+1}}$?

(A) $\frac{5}{4}$ **(B)** $\frac{3}{2}$ **(C)** $\frac{5}{3}$ **(D)** 2 **(E)** 3

3. What number is one third of the way from $\frac{1}{4}$ to $\frac{3}{4}$?

(A) $\frac{1}{3}$ **(B)** $\frac{5}{12}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{7}{12}$ **(E)** $\frac{2}{3}$

4. Four coins are picked out of a piggy bank that contains a collection of pennies, nickels, dimes, and quarters. Which of the following could *not* be the total value of the four coins, in cents?

(A) 15 **(B)** 25 **(C)** 35 **(D)** 45 **(E)** 55

5. One dimension of a cube is increased by 1, another is decreased by 1, and the third is left unchanged. The volume of the new rectangular solid is 5 less than that of the cube. What was the volume of the cube?

(A) 8 **(B)** 27 **(C)** 64 **(D)** 125 **(E)** 216

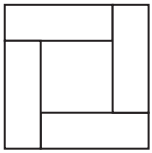
6. Suppose that $P = 2^m$ and $Q = 3^n$. Which of the following is equal to 12^{mn} for every pair of integers (m, n)?

(A) P^2Q **(B)** P^nQ^m **(C)** P^nQ^{2m} **(D)** $P^{2m}Q^n$ **(E)** $P^{2n}Q^m$

7. The first three terms of an arithmetic sequence are $2x - 3$, $5x - 11$, and $3x + 1$ respectively. The nth term of the sequence is 2009. What is n?

(A) 255 **(B)** 502 **(C)** 1004 **(D)** 1506 **(E)** 8037

8. Four congruent rectangles are placed as shown. The area of the outer square is 4 times that of the inner square. What is the ratio of the length of the longer side of each rectangle to the length of its shorter side?

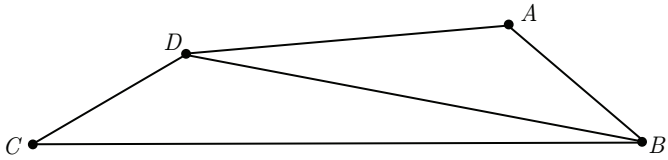


(A) 3 **(B)** $\sqrt{10}$ **(C)** $2+\sqrt{2}$ **(D)** $2\sqrt{3}$ **(E)** 4

9. Suppose that $f(x+3) = 3x^2+7x+4$ and $f(x) = ax^2+bx+c$. What is $a+b+c$?

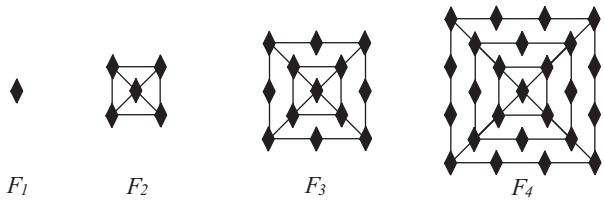
(A) -1 **(B)** 0 **(C)** 1 **(D)** 2 **(E)** 3

10. In quadrilateral $ABCD$, $AB = 5$, $BC = 17$, $CD = 5$, $DA = 9$, and BD is an integer. What is BD?



(A) 11 **(B)** 12 **(C)** 13 **(D)** 14 **(E)** 15

11. The figures F_1, F_2, F_3 and F_4 shown are the first in a sequence of figures. For $n \geq 3$, F_n is constructed from F_{n-1} by surrounding it with a square and placing one more diamond on each side of the new square than F_{n-1} had on each side of its outside square. For example, figure F_3 has 13 diamonds. How many diamonds are there in figure F_{20}?

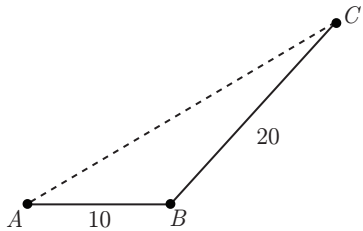


(A) 401 **(B)** 485 **(C)** 585 **(D)** 626 **(E)** 761

12. How many positive integers less than 1000 are 6 times the sum of their digits?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 4 **(E)** 12

13. A ship sails 10 miles in a straight line from A to B, turns through an angle between $45°$ and $60°$, and then sails another 20 miles to C. Let AC be measured in miles. Which of the following intervals contains AC^2?



(A) $[400, 500]$ **(B)** $[500, 600]$ **(C)** $[600, 700]$ **(D)** $[700, 800]$
(E) $[800, 900]$

14. A triangle has vertices $(0, 0)$, $(1, 1)$, and $(6m, 0)$, and the line $y = mx$ divides the triangle into two triangles of equal area. What is the sum of all possible values of m?

(A) $-\frac{1}{3}$ **(B)** $-\frac{1}{6}$ **(C)** $\frac{1}{6}$ **(D)** $\frac{1}{3}$ **(E)** $\frac{1}{2}$

15. For what value of n is $i + 2i^2 + 3i^3 + \cdots + ni^n = 48 + 49i$?
Note: here $i = \sqrt{-1}$.

(A) 24 **(B)** 48 **(C)** 49 **(D)** 97 **(E)** 98

16. A circle with center C is tangent to the positive x- and y-axes and externally tangent to the circle centered at $(3, 0)$ with radius 1. What is the sum of all possible radii of the circle with center C?

(A) 3 **(B)** 4 **(C)** 6 **(D)** 8 **(E)** 9

17. Let $a + ar_1 + ar_1^2 + ar_1^3 + \cdots$ and $a + ar_2 + ar_2^2 + ar_2^3 + \cdots$ be two different infinite geometric series of positive numbers with the same first term. The sum of the first series is r_1, and the sum of the second series is r_2. What is $r_1 + r_2$?

(A) 0 **(B)** $\frac{1}{2}$ **(C)** 1 **(D)** $\frac{1+\sqrt{5}}{2}$ **(E)** 2

18. For $k > 0$, let $I_k = 10\ldots064$, where there are k zeros between the 1 and the 6. Let $N(k)$ be the number of factors of 2 in the prime factorization of I_k. What is the maximum value of $N(k)$?

(A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

19. Andrea inscribed a circle inside a regular pentagon, circumscribed a circle around the pentagon, and calculated the area of the region between the two circles. Bethany did the same with a regular heptagon (7 sides). The areas of the two regions were A and B, respectively. Each polygon had a side length of 2. Which of the following is true?

(A) $A = \frac{25}{49}B$ **(B)** $A = \frac{5}{7}B$ **(C)** $A = B$ **(D)** $A = \frac{7}{5}B$ **(E)** $A = \frac{49}{25}B$

20. Convex quadrilateral $ABCD$ has $AB = 9$ and $CD = 12$. Diagonals $\overline{AC}$ and $\overline{BD}$ intersect at E, $AC = 14$, and $\triangle AED$ and $\triangle BEC$ have equal areas. What is AE?

(A) $\frac{9}{2}$ **(B)** $\frac{50}{11}$ **(C)** $\frac{21}{4}$ **(D)** $\frac{17}{3}$ **(E)** 6

21. Let $p(x) = x^3 + ax^2 + bx + c$, where a, b, and c are complex numbers. Suppose that

$$p(2009 + 9002\pi i) = p(2009) = p(9002) = 0.$$

What is the number of nonreal zeros of $x^{12} + ax^8 + bx^4 + c$?

(A) 4 **(B)** 6 **(C)** 8 **(D)** 10 **(E)** 12

22. A regular octahedron has side length 1. A plane parallel to two of its opposite faces cuts the octahedron into two congruent solids. The polygon formed by the intersection of the plane and the octahedron has area $\frac{a\sqrt{b}}{c}$, where a, b, and c are positive integers, a and c are relatively prime, and b is not divisible by the square of any prime. What is $a + b + c$?

(A) 10 **(B)** 11 **(C)** 12 **(D)** 13 **(E)** 14

23. Functions f and g are quadratic, $g(x) = -f(100 - x)$, and the graph of g contains the vertex of the graph of f. The four x-intercepts on the two graphs have x-coordinates x_1, x_2, x_3, and x_4, in increasing order, and $x_3 - x_2 = 150$. The value of $x_4 - x_1$ is $m + n\sqrt{p}$, where m, n, and p are positive integers, and p is not divisible by the square of any prime. What is $m + n + p$?

(A) 602 **(B)** 652 **(C)** 702 **(D)** 752 **(E)** 802

24. The *tower function of twos* is defined recursively as follows: $T(1) = 2$ and $T(n+1) = 2^{T(n)}$ for $n \geq 1$. Let $A = (T(2009))^{T(2009)}$ and $B = (T(2009))^{A}$. What is the largest integer k such that

$$\underbrace{\log_2 \log_2 \log_2 \ldots \log_2}_{k \text{ times}} B$$

is defined?

(A) 2009 **(B)** 2010 **(C)** 2011 **(D)** 2012 **(E)** 2013

25. The first two terms of a sequence are $a_1 = 1$ and $a_2 = \frac{1}{\sqrt{3}}$. For $n \geq 1$,

$$a_{n+2} = \frac{a_n + a_{n+1}}{1 - a_n a_{n+1}}.$$

What is $|a_{2009}|$?

(A) 0 **(B)** $2 - \sqrt{3}$ **(C)** $\dfrac{1}{\sqrt{3}}$ **(D)** 1 **(E)** $2 + \sqrt{3}$

WRITE TO US!

Correspondence about the problems and solutions for this AMC 12 and orders for publications should be addressed to:

American Mathematics Competitions
University of Nebraska, P.O. Box 81606
Lincoln, NE 68501-1606
Phone: 402-472-2257; Fax: 402-472-6087; email: amcinfo@maa.org

The problems and solutions for this AMC 12 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 12 Subcommittee Chair:

Prof. Bernardo Abrego, Dept. of Mathematics,
California State University-Northridge, Northridge, CA 91330-8313
bernardo.abrego@csun.edu

2009 AIME

The 27th annual AIME will be held on Tuesday, March 17, with the alternate on Wednesday, April 1. It is a 15-question, 3-hour, integer-answer exam. You will be invited to participate only if you score 120 or above or finish in the top 1% of the AMC 10, or if you score 100 or above or finish in the top 5% of the AMC 12. Top-scoring students on the AMC 10/12/AIME will be selected to take the USA Mathematical Olympiad (USAMO) on April 28 - 29, 2009. The best way to prepare for the AIME and USAMO is to study previous exams. Copies may be ordered as indicated below.

PUBLICATIONS

A complete listing of current publications, with ordering instructions, is at our web site: www.unl.edu/amc.

2009

AMC 12 – CONTEST A

DO NOT OPEN UNTIL

TUESDAY, February 10, 2009

****Administration On An Earlier Date Will Disqualify Your School's Results****

1. All information (Rules and Instructions) needed to administer this exam is contained in the TEACHERS' MANUAL, which is outside of this package. **PLEASE READ THE MANUAL BEFORE February 10, 2009.** Nothing is needed from inside this package until February 10.
2. Your PRINCIPAL or VICE PRINCIPAL must sign the Certification Form found in the Teachers' Manual.
3. The Answer Forms must be mailed by First Class mail to the AMC no later than 24 hours following the examination.
4. *The publication, reproduction or communication of the problems or solutions of this test during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination during this period via copier, telephone, email, World Wide Web or media of any type is a violation of the competition rules.*

The American Mathematics Competitions

are Sponsored by

The Mathematical Association of America — MAA ..... www.maa.org/
The Akamai Foundation – ..... www.akamai.com/

Contributors

Academy of Applied Sciences — AAS ..... www.aas-world.org/
American Mathematical Association of Two-Year Colleges — AMATYC ..... www.amatyc.org/
American Mathematical Society — AMS ..... www.ams.org/
American Society of Pension Actuaries — ASPA ..... www.aspa.org/
American Statistical Association — ASA ..... www.amstat.org/
Art of Problem Solving — AoPS ..... www.artofproblemsolving.com/
Awesome Math ..... www.awesomemath.org/
Canada/USA Mathcamp — C/USA MC ..... www.mathcamp.org/
Casualty Actuarial Society — CAS ..... www.casact.org/
Clay Mathematics Institute — CMI ..... www.claymath.org/
IDEA Math ..... www.ideamath.org/
Institute for Operations Research and the Management Sciences — INFORMS ..... www.informs.org/
L. G. Balfour Company ..... www.balfour.com/
Math Zoom Academy ..... www.mathzoom.org/
Mu Alpha Theta — MAT ..... www.mualphatheta.org/
National Assessment & Testing ..... www.natassessment.com/
National Council of Teachers of Mathematics — NCTM ..... www.nctm.org/
Pi Mu Epsilon — PME ..... www.pme-math.org/
Society of Actuaries — SOA ..... www.soa.org/
U. S. A. Math Talent Search — USAMTS ..... www.usamts.org/
W. H. Freeman and Company ..... www. whfreeman.com/
Wolfram Research Inc. ..... www.wolfram.com/