Wednesday, FEBRUARY 25, 2009

60th Annual American Mathematics Contest 12

AMC 12
CONTEST B

THE MATHEMATICAL ASSOCIATION OF AMERICA
American Mathematics Competitions

1. DO NOT OPEN THIS BOOKLET UNTIL YOUR PROCTOR GIVES THE SIGNAL TO BEGIN.
2. This is a 25-question, multiple choice test. Each question is followed by answers marked A, B, C, D and E. Only one of these is correct.
3. Mark your answer to each problem on the AMC 12 Answer Form with a #2 pencil. Check the blackened circles for accuracy and erase errors and stray marks completely. Only answers properly marked on the answer form will be graded.
4. SCORING: You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. No aids are permitted other than scratch paper, graph paper, ruler, compass, protractor, and erasers. No calculators are allowed. No problems on the test will *require* the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the test, your proctor will ask you to record certain information on the answer form. When your proctor gives the signal, begin working the problems. You will have **75 MINUTES** to complete the test.
8. When you finish the exam, *sign your name* in the space provided on the Answer Form.

Students who score 100 or above or finish in the top 5% on this AMC 12 will be invited to take the 27th annual American Invitational Mathematics Examination (AIME) on Tuesday, March 17, 2009 or Wednesday, April 1, 2009. More details about the AIME and other information are on the back page of this test booklet.

The Committee on the American Mathematics Competitions (CAMC) reserves the right to re-examine students before deciding whether to grant official status to their scores. The CAMC also reserves the right to disqualify all scores from a school if it is determined that the required security procedures were not followed.

1. Each morning of her five-day workweek, Jane bought either a 50-cent muffin or a 75-cent bagel. Her total cost for the week was a whole number of dollars. How many bagels did she buy?

   (A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

2. Paula the painter had just enough paint for 30 identically sized rooms. Unfortunately, on the way to work, three cans of paint fell off her truck, so she had only enough paint for 25 rooms. How many cans of paint did she use for the 25 rooms?

   (A) 10 **(B)** 12 **(C)** 15 **(D)** 18 **(E)** 25

3. Twenty percent less than 60 is one-third more than what number?

   (A) 16 **(B)** 30 **(C)** 32 **(D)** 36 **(E)** 48

4. A rectangular yard contains two flower beds in the shape of congruent isosceles right triangles. The remainder of the yard has a trapezoidal shape, as shown. The parallel sides of the trapezoid have lengths 15 and 25 meters. What fraction of the yard is occupied by the flower beds?

   (A) $\frac{1}{8}$ **(B)** $\frac{1}{6}$ **(C)** $\frac{1}{5}$ **(D)** $\frac{1}{4}$ **(E)** $\frac{1}{3}$

5. Kiana has two older twin brothers. The product of their three ages is 128. What is the sum of their three ages?

   (A) 10 **(B)** 12 **(C)** 16 **(D)** 18 **(E)** 24

6. By inserting parentheses, it is possible to give the expression

$$2 \times 3 + 4 \times 5$$

   several values. How many different values can be obtained?

   (A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

7. In a certain year the price of gasoline rose by 20% during January, fell by 20% during February, rose by 25% during March, and fell by x% during April. The price of gasoline at the end of April was the same as it had been at the beginning of January. To the nearest integer, what is x?

(A) 12 **(B)** 17 **(C)** 20 **(D)** 25 **(E)** 35

8. When a bucket is two-thirds full of water, the bucket and water weigh a kilograms. When the bucket is one-half full of water the total weight is b kilograms. In terms of a and b, what is the total weight in kilograms when the bucket is full of water?

(A) $\frac{2}{3}a + \frac{1}{3}b$ **(B)** $\frac{3}{2}a - \frac{1}{2}b$ **(C)** $\frac{3}{2}a + b$

(D) $\frac{3}{2}a + 2b$ **(E)** $3a - 2b$

9. Triangle ABC has vertices $A = (3, 0)$, $B = (0, 3)$, and C, where C is on the line $x + y = 7$. What is the area of $\triangle ABC$?

(A) 6 **(B)** 8 **(C)** 10 **(D)** 12 **(E)** 14

10. A particular 12-hour digital clock displays the hour and minute of a day. Unfortunately, whenever it is supposed to display a 1, it mistakenly displays a 9. For example, when it is 1:16 PM the clock incorrectly shows 9:96 PM. What fraction of the day will the clock show the correct time?

(A) $\frac{1}{2}$ **(B)** $\frac{5}{8}$ **(C)** $\frac{3}{4}$ **(D)** $\frac{5}{6}$ **(E)** $\frac{9}{10}$

11. On Monday, Millie puts a quart of seeds, 25% of which are millet, into a bird feeder. On each successive day she adds another quart of the same mix of seeds without removing any seeds that are left. Each day the birds eat only 25% of the millet in the feeder, but they eat all of the other seeds. On which day, just after Millie has placed the seeds, will the birds find that more than half the seeds in the feeder are millet?

(A) Tuesday **(B)** Wednesday **(C)** Thursday **(D)** Friday

(E) Saturday

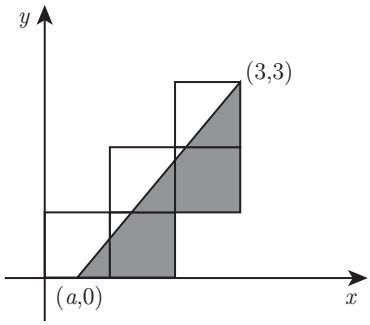
12. The fifth and eighth terms of a geometric sequence of real numbers are 7! and 8! respectively. What is the first term?

(A) 60 **(B)** 75 **(C)** 120 **(D)** 225 **(E)** 315

13. Triangle ABC has $AB = 13$ and $AC = 15$, and the altitude to $\overline{BC}$ has length 12. What is the sum of the two possible values of BC?

(A) 15 **(B)** 16 **(C)** 17 **(D)** 18 **(E)** 19

14. Five unit squares are arranged in the coordinate plane as shown, with the lower left corner at the origin. The slanted line, extending from $(a, 0)$ to $(3, 3)$, divides the entire region into two regions of equal area. What is a?



(A) $\frac{1}{2}$ **(B)** $\frac{3}{5}$ **(C)** $\frac{2}{3}$ **(D)** $\frac{3}{4}$ **(E)** $\frac{4}{5}$

15. Assume $0 < r < 3$. Below are five equations for x. Which equation has the largest solution x?

(A) $3(1 + r)^x = 7$ **(B)** $3(1 + r/10)^x = 7$ **(C)** $3(1 + 2r)^x = 7$

(D) $3(1 + \sqrt{r})^x = 7$ **(E)** $3(1 + 1/r)^x = 7$

16. Trapezoid $ABCD$ has $\overline{AD} \parallel \overline{BC}$, $BD = 1$, $\angle DBA = 23°$, and $\angle BDC = 46°$. The ratio $BC : AD$ is $9 : 5$. What is CD?

(A) $\frac{7}{9}$ **(B)** $\frac{4}{5}$ **(C)** $\frac{13}{15}$ **(D)** $\frac{8}{9}$ **(E)** $\frac{14}{15}$

17. Each face of a cube is given a single narrow stripe painted from the center of one edge to the center of its opposite edge. The choice of the edge pairing is made at random and independently for each face. What is the probability that there is a continuous stripe encircling the cube?

(A) 1/8 **(B)** 3/16 **(C)** 1/4 **(D)** 3/8 **(E)** 1/2

18. Rachel and Robert run on a circular track. Rachel runs counterclockwise and completes a lap every 90 seconds, and Robert runs clockwise and completes a lap every 80 seconds. Both start from the start line at the same time. At some random time between 10 minutes and 11 minutes after they begin to run, a photographer standing inside the track takes a picture that shows one-fourth of the track, centered on the starting line. What is the probability that both Rachel and Robert are in the picture?

(A) $\frac{1}{16}$ **(B)** $\frac{1}{8}$ **(C)** $\frac{3}{16}$ **(D)** $\frac{1}{4}$ **(E)** $\frac{5}{16}$

19. For each positive integer n, let $f(n) = n^4 - 360n^2 + 400$. What is the sum of all values of $f(n)$ that are prime numbers?

(A) 794 **(B)** 796 **(C)** 798 **(D)** 800 **(E)** 802

20. A convex polyhedron Q has vertices $V_1, V_2, \ldots, V_n$, and 100 edges. The polyhedron is cut by planes $P_1, P_2, \ldots, P_n$ in such a way that plane P_k cuts only those edges that meet at vertex V_k. In addition, no two planes intersect inside or on Q. The cuts produce n pyramids and a new polyhedron R. How many edges does R have?

(A) 200 **(B)** $2n$ **(C)** 300 **(D)** 400 **(E)** $4n$

21. Ten women sit in 10 seats in a line. All of the 10 get up and then reseat themselves using all 10 seats, each sitting in the seat she was in before or a seat next to the one she occupied before. In how many ways can the women be reseated?

(A) 89 **(B)** 90 **(C)** 120 **(D)** 2^{10} **(E)** $2^2 3^8$

22. Parallelogram $ABCD$ has area 1,000,000. Vertex A is at $(0,0)$ and all other vertices are in the first quadrant. Vertices B and D are lattice points on the lines $y = x$ and $y = kx$ for some integer $k > 1$, respectively. How many such parallelograms are there?

(A) 49 **(B)** 720 **(C)** 784 **(D)** 2009 **(E)** 2048

23. A region S in the complex plane is defined by

$$S = \{x + iy : -1 \le x \le 1, -1 \le y \le 1\}.$$

A complex number $z = x + iy$ is chosen uniformly at random from S. What is the probability that $(\frac{3}{4} + \frac{3}{4}i)z$ is also in S?

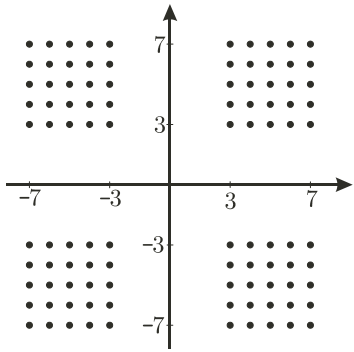
(A) $\frac{1}{2}$ **(B)** $\frac{2}{3}$ **(C)** $\frac{3}{4}$ **(D)** $\frac{7}{9}$ **(E)** $\frac{7}{8}$

24. For how many values of x in $[0, \pi]$ is $\sin^{-1}(\sin 6x) = \cos^{-1}(\cos x)$?

Note: The functions $\sin^{-1} = \arcsin$ and $\cos^{-1} = \arccos$ denote inverse trigonometric functions.

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

25. The set G is defined by the points (x, y) with integer coordinates, $3 \le |x| \le 7$, and $3 \le |y| \le 7$. How many squares of side at least 6 have their four vertices in G?



(A) 125 **(B)** 150 **(C)** 175 **(D)** 200 **(E)** 225

WRITE TO US!

Correspondence about the problems and solutions for this AMC 12 and orders for publications should be addressed to:

American Mathematics Competitions
University of Nebraska, P.O. Box 81606
Lincoln, NE 68501-1606
Phone: 402-472-2257; Fax: 402-472-6087; email: amcinfo@maa.org

The problems and solutions for this AMC 12 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 12 Subcommittee Chair:

Prof. Bernardo Abrego, Dept. of Mathematics
California State University-Northridge, Northridge, CA 91330-8313
bernardo.abrego@csun.edu

2009 AIME

The 27th annual AIME will be held on Tuesday, March 17, with the alternate on Wednesday, April 1. It is a 15-question, 3-hour, integer-answer exam. You will be invited to participate only if you score 120 or above or finish in the top 1% of the AMC 10, or if you score 100 or above or finish in the top 5% of the AMC 12. Top-scoring students on the AMC 10/12/AIME will be selected to take the USA Mathematical Olympiad (USAMO) on April 28 - 29, 2009. The best way to prepare for the AIME and USAMO is to study previous exams. Copies may be ordered as indicated below.

PUBLICATIONS

A complete listing of current publications, with ordering instructions, is at our web site: www.unl.edu/amc.

2009

AMC 12 – CONTEST B

DO NOT OPEN UNTIL

WEDNESDAY, February 25, 2009

****Administration On An Earlier Date Will Disqualify Your School's Results****

1. All information (Rules and Instructions) needed to administer this exam is contained in the TEACHERS' MANUAL, which is outside of this package. **PLEASE READ THE MANUAL BEFORE February 25, 2009.** Nothing is needed from inside this package until February 25.
2. Your PRINCIPAL or VICE PRINCIPAL must sign the Certification Form found in the Teachers' Manual.
3. The Answer Forms must be mailed by First Class mail to the AMC no later than 24 hours following the examination.
4. *The publication, reproduction or communication of the problems or solutions of this test during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination during this period via copier, telephone, email, World Wide Web or media of any type is a violation of the competition rules.*

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