



AMERICAN MATHEMATICS COMPETITIONS

51<sup>st</sup> Annual American Mathematics Contest → 12

**AMC → 12**

**TUESDAY, FEBRUARY 15, 2000**

Sponsored by

Mathematical Association of America

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Clay Mathematics Institute University of Nebraska-Lincoln

1. DO NOT OPEN THIS BOOKLET UNTIL TOLD TO DO SO BY YOUR PROCTOR.
2. This is a twenty-five question, multiple choice test. Each question is followed by answers marked A,B,C,D and E. Only one of these is correct.
3. The answers to the problems are to be marked on the AMC → 12 Answer Form with a #2 pencil. Check the blackened circles for accuracy and erase errors and stray marks completely. Only answers properly marked on the answer form will be graded.
4. SCORING: You will receive 6 points for each correct answer, 2 points for each problem left unanswered, and 0 points for each incorrect answer.
5. No aids are permitted other than scratch paper, graph paper, ruler, compass, protractor, erasers and calculators that are accepted for use on the SAT. No problems on the test will require the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the test, your proctor will ask you to record certain information on the answer form. When your proctor gives the signal, begin working the problems. You will have **75 MINUTES** working time to complete the test.
8. When you finish the exam, *sign your name* in the space provided on the Answer Form.

*Students who score 100 or above on this AMC → 12 will be invited to take the 18th annual American Invitational Mathematics Examination (AIME) on Tuesday, March 28, 2000. More details about the AIME and other information are on the back page of this test booklet.*

The Committee on the American Mathematics Competitions (CAMC) reserves the right to re-examine students before deciding whether to grant official status to their scores. The CAMC also reserves the right to disqualify all scores from a school if it is determined that the required security procedures were not followed.

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1. In the year 2001, the United States will host the International Mathematical Olympiad. Let  $I$ ,  $M$ , and  $O$  be distinct positive integers such that the product  $I \cdot M \cdot O = 2001$ . What is the largest possible value of the sum  $I + M + O$ ?
- (A) 23      (B) 55      (C) 99      (D) 111      (E) 671
2.  $2000(2000^{2000}) =$
- (A)  $2000^{2001}$       (B)  $4000^{2000}$       (C)  $2000^{4000}$   
(D)  $4,000,000^{2000}$       (E)  $2000^{4,000,000}$
3. Each day, Jenny ate 20% of the jellybeans that were in her jar at the beginning of that day. At the end of second day, 32 remained. How many jellybeans were in the jar originally?
- (A) 40      (B) 50      (C) 55      (D) 60      (E) 75
4. The Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, 21, ... starts with two 1s, and each term afterwards is the sum of its two predecessors. Which one of the ten digits is the last to appear in the units position of a number in the Fibonacci sequence?
- (A) 0      (B) 4      (C) 6      (D) 7      (E) 9
5. If  $|x - 2| = p$ , where  $x < 2$ , then  $x - p =$
- (A)  $-2$       (B) 2      (C)  $2 - 2p$       (D)  $2p - 2$       (E)  $|2p - 2|$
6. Two different prime numbers between 4 and 18 are chosen. When their sum is subtracted from their product, which of the following number could be obtained?
- (A) 21      (B) 60      (C) 119      (D) 180      (E) 231
7. How many positive integers  $b$  have the property that  $\log_b 729$  is a positive integer?
- (A) 0      (B) 1      (C) 2      (D) 3      (E) 4

8. Figure 0, 1, 2, and 3 consist of 1, 5, 13, and 25 nonoverlapping unit squares, respectively. If the pattern were continued, how many nonoverlapping unit squares would there be in figure 100?

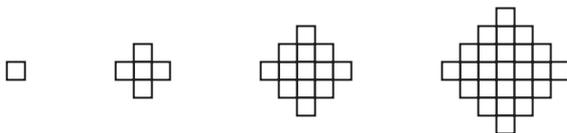


figure 0

figure 1

figure 2

figure 3

- (A) 10401      (B) 19801      (C) 20201      (D) 39801      (E) 40801
9. Mrs. Walter gave an exam in a mathematics class of five students. She entered the scores in random order into a spreadsheet, which recalculated the class average after each score was entered. Mrs. Walter noticed that after each score was entered, the average was always an integer. The scores (listed in ascending order) were 71, 76, 80, 82, and 91. What was the last scores Mrs. Walter entered?
- (A) 71      (B) 76      (C) 80      (D) 82      (E) 91
10. The point  $P = (1, 2, 3)$  is reflected in the  $xy$ -plane, then its image  $Q$  is rotated by  $180^\circ$  about the  $x$ -axis to produce  $R$ , and finally,  $R$  is translated by 5 units in the positive- $y$  direction to produce  $S$ . What are the coordinates of  $S$ ?
- (A)  $(1, 7, -3)$       (B)  $(-1, 7, -3)$       (C)  $(-1, -2, 8)$   
 (D)  $(-1, 3, 3)$       (E)  $(1, 3, 3)$
11. Two non-zero real numbers,  $a$  and  $b$ , satisfy  $ab = a - b$ . Find a possible value of  $\frac{a}{b} + \frac{b}{a} - ab$ .
- (A)  $-2$       (B)  $-\frac{1}{2}$       (C)  $\frac{1}{3}$       (D)  $\frac{1}{2}$       (E)  $2$
12. Let  $A$ ,  $M$ , and  $C$  be nonnegative integers such that  $A + M + C = 12$ . What is the maximum value of  $A \cdot M \cdot C + A \cdot M + M \cdot C + C \cdot A$ ?
- (A) 62      (B) 72      (C) 92      (D) 102      (E) 112
13. One morning each member of Angela's family drank an 8-ounce mixture of coffee with milk. The amounts of coffee and milk varied from cup to cup, but were never zero. Angela drank a quarter of the total amount of milk and a sixth of the total amount of coffee. How many people are in the family?
- (A) 3      (B) 4      (C) 5      (D) 6      (E) 7

14. When the mean, median, and mode of the list

$$10, 2, 5, 2, 4, 2, x$$

are arranged in increasing order, they form a non-constant arithmetic progression. What is the sum of all possible real value of  $x$ ?

- (A) 3    (B) 6    (C) 9    (D) 17    (E) 20

15. Let  $f$  be a function for which  $f(x/3) = x^2 + x + 1$ . Find the sum of all values of  $z$  for which  $f(3z) = 7$ .

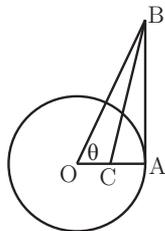
- (A)  $-1/3$     (B)  $-1/9$     (C) 0    (D)  $5/9$     (E)  $5/3$

16. A checkerboard of 13 rows and 17 columns has a number written in each square, beginning in the upper left corner, so that the first row is numbered 1, 2, ..., 17, the second row 18, 19, ..., 34, and so on down the board. If the board is renumbered so that the left column, top to bottom, is 1, 2, ..., 13, the second column 14, 15, ..., 26 and so on across the board, some square have the same numbers in both numbering systems. Find the sum of the numbers in these squares (under either system).

- (A) 222    (B) 333    (C) 444    (D) 555    (E) 666

17. A circle centered at  $O$  has radius 1 and contains the point  $A$ . Segment  $AB$  is tangent to the circle at  $A$  and  $\angle AOB = \theta$ . If point  $C$  lies on  $\overline{OA}$  and  $\overline{BC}$  bisects  $\angle ABO$ , then  $OC =$

- (A)  $\sec^2 \theta - \tan \theta$     (B)  $\frac{1}{2}$     (C)  $\frac{\cos^2 \theta}{1 + \sin \theta}$   
(D)  $\frac{1}{1 + \sin \theta}$     (E)  $\frac{\sin \theta}{\cos^2 \theta}$



18. In year  $N$ , the 300<sup>th</sup> day of the year is a Tuesday. In year  $N + 1$ , the 200<sup>th</sup> day is also a Tuesday. On what day of the week did the 100<sup>th</sup> day of year  $N - 1$  occur?

- (A) Thursday    (B) Friday    (C) Saturday    (D) Sunday    (E) Monday

19. In triangle  $ABC$ ,  $AB = 13$ ,  $BC = 14$ , and  $AC = 15$ . Let  $D$  denote the midpoint of  $\overline{BC}$  and let  $E$  denote the intersection of  $\overline{BC}$  with the bisector of angle  $BAC$ . Which of the following is closest to the area of the triangle  $ADE$ ?

(A) 2      (B) 2.5      (C) 3      (D) 3.5      (E) 4

20. If  $x$ ,  $y$ , and  $z$  are positive numbers satisfying

$$x + 1/y = 4, \quad y + 1/z = 1, \quad \text{and} \quad z + 1/x = 7/3,$$

then  $xyz =$

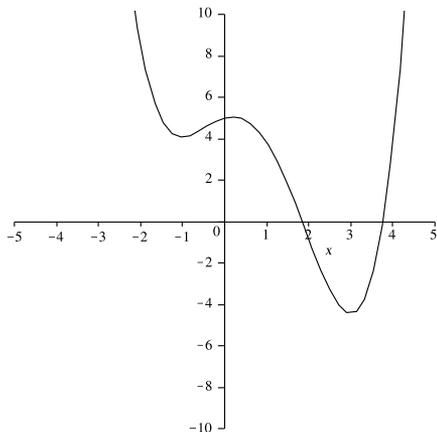
(A)  $2/3$       (B) 1      (C)  $4/3$       (D) 2      (E)  $7/3$

21. Through a point on the hypotenuse of a right triangle, lines are drawn parallel to the legs of the triangle so that the triangle is divided into a square and two smaller right triangles. The area of one of the two small right triangle is  $m$  times the area of the square. The ratio of the area of the other small right triangle to the area of the square is

(A)  $\frac{1}{2m+1}$       (B)  $m$       (C)  $1-m$       (D)  $\frac{1}{4m}$       (E)  $\frac{1}{8m^2}$

22. The graph below shows a portion of the curve defined by the quartic polynomial  $P(x) = x^4 + ax^3 + bx^2 + cx + d$ . Which of the following is the smallest?

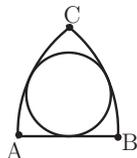
- (A)  $P(-1)$   
(B) The product of the zeros of  $P$   
(C) The product of the non-real zeros of  $P$   
(D) The sum of the coefficients of  $P$   
(E) The sum of the real zeros of  $P$



23. Professor Gamble buys a lottery ticket, which requires that he pick six different integers from 1 through 46, inclusive. He chooses his numbers so that the sum of the base-ten logarithms of his six numbers is an integer. It so happens that the integers on the winning ticket have the same property— the sum of the base-ten logarithms is an integer. What is the probability that Professor Gamble holds the winner ticket?

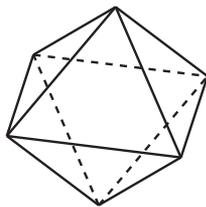
(A)  $1/5$     (B)  $1/4$     (C)  $1/3$     (D)  $1/2$     (E) 1

24. If circular arcs  $AC$  and  $BC$  have centers at  $B$  and  $A$ , respectively, then there exists a circle tangent to both  $\widehat{AC}$  and  $\widehat{BC}$ , and to  $\overline{AB}$ . If the length of  $\widehat{BC}$  is 12, then the circumference of the circle is



(A) 24    (B) 25    (C) 26    (D) 27    (E) 28

25. Eight congruent equilateral triangles, each of a different color, are used to construct a regular octahedron. How many distinguishable ways are there to construct the octahedron? (Two colored octahedrons are distinguishable if neither can be rotated to look just like the other.)



(A) 210    (B) 560    (C) 840  
(D) 1260    (E) 1680

**WRITE TO US!**

Correspondence about the problems and solutions for this AMC  $\rightarrow$  12 should be addressed to:

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Orders for any publications listed below should be addressed to:

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**2000 AIME**

The AIME will be held on Tuesday, March 28, 2000. It is a 15-question, 3-hour, integer-answer exam. You will be invited to participate only if you score in the top 1% of the AMC  $\rightarrow$  10 or receive a score of 100 or above on this AMC  $\rightarrow$  12. Top-scoring students on the AMC  $\rightarrow$  10/12/AIME will be selected to take the USA Mathematical Olympiad (USAMO) on Tuesday, May 2, 2000. The best way to prepare for the AIME and USAMO is to study previous years of these exams. Copies may be ordered as indicated below.

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2000

# AMC → 12



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Your School's Results\*\***

1. All information (Rules and Instructions) needed to administer this exam is contained in the TEACHERS' MANUAL, which is outside of this package. **PLEASE READ THE MANUAL BEFORE FEBRUARY 15.** Nothing is needed from inside this package until February 15.
2. Your PRINCIPAL or VICE PRINCIPAL must sign the Certification Form A found in the Teachers' Manual.
3. The Answer Forms must be mailed by First Class mail to the AMC Director, Titu Andreescu, no later than 24 hours following the examination.
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