# AIME SOLUTIONS PAMPHLET

## FOR STUDENTS AND TEACHERS

## 5th ANNUAL
## AMERICAN INVITATIONAL
## MATHEMATICS EXAMINATION
## 1987

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This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs. geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

It is hoped that teachers will find the time to share these solutions with their students, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution.

AMERICAN MATHEMATICS COMPETITIONS

**AIME Chairman:** **Professor George Berzsenyi**
**Department of Mathematics**
**Lamar University**
**Beaumont, TX 77710 USA**

**Executive Director:** **Professor Walter E. Mientka**
**Department of Mathematics and Statistics**
**University of Nebraska**
**Lincoln, NE 68588-0322 USA**

Correspondence about the Examination questions and solutions should be addressed to the AIME Chairman. Questions about the administrative arrangements, or orders for prior year copies of Examinations given by the Committee, should be addressed to the Executive Director.

1. (Answer: 300)

Since there is no carrying in the addition, the ones column must add to 2, the tens column to 9, the hundreds column to 4 and the thousands column to 1 for each simple ordered pair of non-negative integers summing to 1492. To get a single digit $d$ as the sum of two digits, there are $d+1$ ways:

$$0+d,\ 1+(d-1),\ 2+(d-2),\ \ldots,\ d+0.$$

Thus the number of simple ordered pairs of non-negative integers that sum to 1492 is $(1+1)(4+1)(9+1)(2+1) = 300$.

2. (137)

Let $O$ and $\tilde{O}$ be the centers of the two spheres, and let $P$ and $\tilde{P}$ be the two points where the extensions of the segment $O\tilde{O}$ pierce the two spheres, respectively, so that $O$ is between $P$ and $\tilde{O}$, and $\tilde{O}$ is between $O$ and $\tilde{P}$. Then the desired maximum distance is

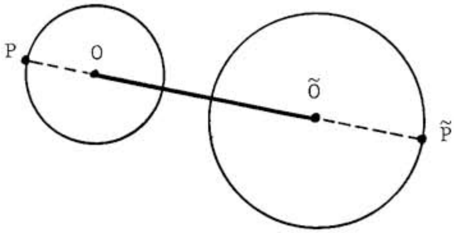
$$P\tilde{P} = PO + O\tilde{O} + \tilde{O}\tilde{P}, \tag{1}$$

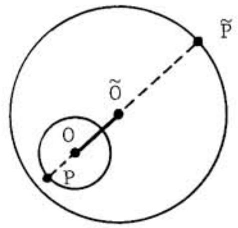
where $PO$ and $\tilde{O}\tilde{P}$ are the given radii and $O\tilde{O}$ is found by the Distance Formula. In our case, $P\tilde{P} = 19+31+87 = 137$.

To see that (1) indeed yields the maximum distance, note that by the Triangle Inequality, for any points $Q$ and $\tilde{Q}$ on the spheres with centers $O$ and $\tilde{O}$, respectively,

$$Q\tilde{Q} \le QO + O\tilde{Q} \le QO + O\tilde{O} + \tilde{O}\tilde{Q} = PO + O\tilde{O} + \tilde{O}\tilde{P} = P\tilde{P}.$$

Note. The above solution does not depend on the position of the spheres relative to one another, as can be seen in the two configurations below, showing cross sections of the spheres by planes containing their centers.



3. (182)

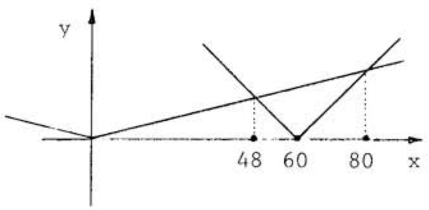
Let $k$ be a positive integer, and let $1, d_1, d_2, \ldots, d_{n-1}, d_n, k$ be its divisors in ascending order. Then $1 \cdot k = d_1 \cdot d_n = d_2 \cdot d_{n-1} = \cdots$. For $k$ to be nice, we must have $n = 2$. Moreover, $d_1$ must be prime, for otherwise, the proper divisors of $d_1$ would have appeared in the listing above between 1 and $d_1$. Similarly, $d_2$ is either a prime or the square of $d_1$, for otherwise $d_1$ could not be the only divisor between 1 and $d_2$. Therefore, $k$ is either the product of two distinct primes or -- being the product of a prime and its square -- is the cube of a prime.

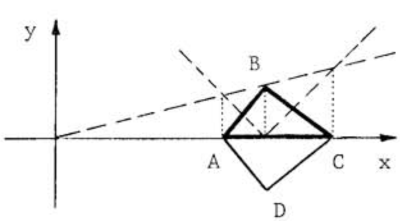
In view of the above, one can easily list the first ten nice numbers. They are: 6, 8, 10, 14, 15, 21, 22, 26, 27 and 33. Their sum is 182.

4. (480)

First we note that the graph of the given equation is symmetric with respect to the x-axis, since the replacement of $y$ by $-y$ does not change the equation. Consequently, it suffices to assume that $y \geq 0$, find the area enclosed above the x-axis, and then double this area to find the answer to the problem.

By sketching both $y = |x/4|$ and $y = |x - 60|$ on the same graph, as shown in the first figure below, we note that $y = |x/4| - |x - 60| \geq 0$ only if the graph of $y = |x/4|$ lies above that of $y = |x - 60|$. This occurs only between $x = 48$ (where $x/4 = 60 - x$) and $x = 80$ (where $x/4 = x - 60$). In this interval, the graph of $y = |x/4| - |x - 60|$ with the x-axis forms $\Delta ABC$, as shown in the second figure. The base of this triangle is $80 - 48 = 32$, its altitude (at $x = 60$) is 15, hence its area is 240. Kite ABCD is the graph of $|x - 60| + |y| = |x/4|$; the area enclosed by it is $2 \cdot 240 = 480$.



5. (588)

Rewrite the given equation in the form

$$(y^2 - 10)(3x^2 + 1) = 3\cdot 13^2,$$

and note that, since y is an integer and $3x^2+1$ is a positive integer, $y^2-10$ must be a positive integer. Consequently, $y^2-10 = 1, 3, 13, 39, 169$ or $507$, implying that $y^2 = 11, 13, 23, 49, 179$ or $517$. Since the only perfect square in the second list is 49, it follows that $y^2 - 10 = 39$, implying that $3x^2+1 = 13$, $x^2 = 4$ and $3x^2y^2 = 12\cdot 49 = 588$.

6. (193)

Since trapezoids XYQP and ZWPQ have the same area, and since their parallel sides are of the same length, their heights must also be equal, each of length BC/2. Moreover, since XY is one fourth of the perimeter of rectangle ABCD, it follows that $XY = (AB+BC)/2$, and that the area of trapezoid XYQP is $\frac{PQ + [(AB+BC)/2]}{2}\cdot\frac{BC}{2}$. Since this must be equal to one fourth of the area of rectangle ABCD, one finds that

$$\frac{PQ + [(AB+BC)/2]}{2}\cdot\frac{BC}{2} = \frac{AB\cdot BC}{4},$$

from which $AB = BC + 2\cdot PQ = 19 + 2\cdot 87 = 193$ cm.

Note. This problem arose from the following puzzle: how can one divide a rectangular cake, with a thin layer of icing on the top and sides, into n pieces of equal volume, so that each piece has the same amount of icing? The reader may wish to explore such equal-division problems for $n > 4$.

7. (070)

Since both 1000 and 2000 are of the form $2^m5^n$, the numbers a, b and c must also be of this form. More specifically,

(1) $$a = 2^{m_1}5^{n_1}, \qquad b = 2^{m_2}5^{n_2}, \qquad c = 2^{m_3}5^{n_3},$$

where the $m_i$ and $n_i$ are non-negative integers for $i = 1, 2, 3$.

Then, in view of the definition of $[r,s]$, and since

(2) $[a,b] = 2^3 5^3$, $[b,c] = 2^4 5^3$, $[c,a] = 2^4 5^3$,

the following equalities must hold:

(3) $\max\{m_1,m_2\} = 3$, $\max\{m_2,m_3\} = 4$, $\max\{m_3,m_1\} = 4$

and

(4) $\max\{n_1,n_2\} = 3$, $\max\{n_2,n_3\} = 3$, $\max\{n_3,n_1\} = 3$.

To satisfy (3), we must have $m_3 = 4$, and either $m_1$ or $m_2$ must be 3, while the other one can take the values of 0, 1, 2 or 3. There are 7 such ordered triples, namely (0,3,4), (1,3,4), (2,3,4), (3,0,4), (3,1,4), (3,2,4) and (3,3,4).

To satisfy (4), two of $n_1$, $n_2$ and $n_3$ must be 3, while the third one ranges through the values of 0, 1, 2 and 3. The number of such ordered triples is 10; they are (3,3,0), (3,3,1), (3,3,2), (3,0,3), (3,1,3), (3,2,3), (0,3,3), (1,3,3), (2,3,3) and (3,3,3).

Since the choice of $(m_1,m_2,m_3)$ is independent of the choice of $(n_1,n_2,n_3)$, they can be chosen in $7\cdot 10 = 70$ different ways. This is the number of ordered triples $(a,b,c)$ satisfying the given conditions.

8. (112)

By first writing the inequalities in the form $\frac{13}{7} < \frac{n+k}{n} < \frac{15}{8}$, we can see that they are equivalent to

$$48n < 56k < 49n.$$

Consequently, the problem is to find the longest open interval $(48n,49n)$ that contains exactly one integral multiple of 56.

Since the length of the above interval is $n$, it contains $n-1$ integers. If $n-1 \geq 2\cdot 56$, the interval will contain at least two multiples of 56. Hence, $2\cdot 56 = 112$ is the largest candidate for $n$. Indeed, we find that

$$48\cdot 112 = 56\cdot 96 < 56\cdot 97 < 56\cdot 98 = 49\cdot 112,$$

also exhibiting that $k = 97$ is the unique positive integer corresponding to $n = 112$.

9. (033)

Noting that $\measuredangle APB = \measuredangle BPC = \measuredangle CPA = 120°$, and applying the Law of Cosines to $\triangle APB$, $\triangle BPC$ and $\triangle CPA$, we find that

(1) $(AB)^2 = (PA)^2 + (PB)^2 + PA \cdot PB = 100 + 36 + 60 = 196,$

(2) $(BC)^2 = (PB)^2 + (PC)^2 + PB \cdot PC = 36 + (PC)^2 + 6\,PC$

and

(3) $(CA)^2 = (PC)^2 + (PA)^2 + PC \cdot PA = (PC)^2 + 100 + 10\,PC.$

Since $(AB)^2 + (BC)^2 = (CA)^2$ by the Pythagorean Theorem, it follows from (1), (2) and (3) that

$$196 + [36 + (PC)^2 + 6\,PC] = (PC)^2 + 100 + 10\,PC,$$

from which $PC = 33$.

10. (120)

Let $v_1$ denote Al's speed (in steps per unit time) and $t_1$ his time. Similarly, let $v_2$ and $t_2$ denote Bob's speed and time. Moreover, let $v$ be the speed of the escalator, and let $x$ be the number of steps visible at any given time. Then, from the information given,

(1) $v_1 = 3v_2,$ $\quad v_1 t_1 = 150,$ $\quad v_2 t_2 = 75.$

From (1) it follows that

(2) $t_2 / t_1 = 3/2.$

We also know that $x = (v_2 + v)t_2 = (v_1 - v)t_1$, from which we have $v = (x - 75)/t_2 = (150 - x)/t_1$, and hence

(3) $t_2 / t_1 = (x - 75)/(150 - x).$

Therefore, from (2) and (3), upon setting their right sides equal, we find that $x = 120$.

Alternate Solution. Assume that Al and Bob start at the same time from their respective ends of the escalator. Then the number of steps initially separating them is the same as the number of visible steps on the escalator.

Hence, to solve the problem, we must find the number of steps each of them takes until they meet, and then add these two numbers.

Since Al can take $3\cdot 75 = 225$ steps while Bob takes 75 steps, it follows (from $150 = (2/3)\cdot 225$) that Al walks down the escalator in 2/3 of the time it takes Bob to walk up. Therefore, they meet 2/5 of the way from the bottom of the escalator. To that point, Al takes $(3/5)\cdot 150 = 90$ steps, while Bob takes $(2/5)\cdot 75 = 30$ steps. As indicated above, the sum of these, 120, is the number of visible steps of the escalator.

Notes. The second solution illustrates a general way to find the number of exposed steps on a moving escalator: find a friend to start simultaneously at the opposite end, and simply add the number of steps you have each taken before you meet.

Such escalator problems were rather popular at one time. This one was fashioned after a problem of Henry Dudeney (1857-1930), England's most famous creator of puzzles.

11. (486)

To solve the problem, we must find integers n and k such that n is non-negative, k is as large as possible, and

(1) $$3^{11} = (n+1) + (n+2) + \cdots + (n+k).$$

Noting that

$$(n+1) + (n+2) + \cdots + (n+k) = [1+2+\cdots+(n+k)] - [1+2+\cdots+n]$$
$$= \frac{(n+k)(n+k+1)}{2} - \frac{n(n+1)}{2}$$
$$= k(k+2n+1)/2,$$

it follows that (1) is equivalent to

(2) $$k(k+2n+1) = 2\cdot 3^{11}.$$

In solving (2), we must ensure that the smaller factor, k, is as large as possible, and that n is a non-negative integer. These conditions lead to $k = 2\cdot 3^5 = 486$, $n = 121$ and $3^{11} = 122 + 123 + \cdots + 607$.

Alternate Solution. Let m be the average of the k consecutive integers. If k is odd, then m must be the middle integer, and $km = 3^{11}$. Now $k = 3^5$ and $m = 3^6$ is the best we can do, for if $k = 3^6$ then $m - (k-1)/2$, the smallest summand, is negative. But if k is even, then m lies halfway between the middle two integers in the sum. Thus $(2m)k = 2\cdot 3^{11}$ and now the largest even divisor of $2\cdot 3^{11}$ which does not give rise to a negative first summand is $2\cdot 3^5 = 486$. This is the answer.

12. (019)

We solve the equivalent problem of finding the smallest positive integer n for which

(1) $$n^3 + 1 < (n + 10^{-3})^3.$$

This is equivalent to the given problem because

$$n < \sqrt[3]{m} < n + 10^{-3} \iff n^3 < m < (n + 10^{-3})^3,$$

and because if some integer m satisfies the double inequality on the right above, then $n^3 + 1$ is the smallest such m.

Rewriting (1) in the form

(2) $$\frac{1000}{3} < n^2 + \frac{n}{1000} + \frac{1}{3{,}000{,}000},$$

we observe that $n^2$ must be near $1000/3$, for the contributions of the other two terms on the right side of (2) are relatively small. Consequently, since $18^2 < 1000/3 < 19^2$, we expect that either $n = 18$ or $n = 19$. In the first case, (2) is not satisfied; this can be verified by an easy caculation. It is even easier to show that $n = 19$ satisfies (2), so it is the smallest positive integer with the desired property. The corresponding $m = 19^3 + 1 = 6860$ is the smallest positive integer whose cube root has a positive decimal part which is less than $1/1000$.

13. (931)

The key property of bubble pass is that immediately after $r_k$ is compared with its predecessor and possibly switched, the current $r_k$ is the largest member of the set $\{r_1, r_2, \ldots, r_k\}$. Also, this set is the same at this point as it was originally, though the order of its elements in the sequence may be very different. These assertions can be verified by induction.

For a number m, which is initially before $r_{30}$ in the sequence, to end up as $r_{30}$, two things must happen: m must move into the $30^{th}$ position when the current $r_{29}$ (= m) and $r_{30}$ are compared, and it must not move out of that position when compared to $r_{31}$. Therefore, by the key property, m must be the largest number in the original $\{r_1, r_2, \ldots, r_{30}\}$, but not the largest in the original $\{r_1, r_2, \ldots, r_{31}\}$. In other words, of the first 31 numbers originally, the largest must be $r_{31}$ and the second largest must be m, which in our case is $r_{20}$.

Whatever the first 31 numbers are, there are 31! equally likely orderings of them. Of these, 29! have the largest of them in the $31^{st}$ slot and the second largest in the $20^{th}$ slot. (The other 29 numbers have 29! equally likely orderings.) Thus the desired probability must be $\frac{p}{q} = \frac{29!}{31!} = \frac{1}{930}$, and hence $p+q = 931$.

14. (373)

Since $324 = 18^2 = 4\cdot 3^4$, we may use the factorization

$$\begin{aligned}
x^4 + 4y^4 &= x^4 + 4x^2y^2 + 4y^4 - 4x^2y^2 \\
&= (x^2 + 2y^2)^2 - (2xy)^2 \\
&= [(x^2 + 2y^2) - 2xy][(x^2 + 2y^2) + 2xy] \\
&= [(x^2 - 2xy + y^2) + y^2][(x^2 + 2xy + y^2) + y^2] \\
&= [(x - y)^2 + y^2][(x + y)^2 + y^2],
\end{aligned}$$

which yields in our case

$$n^4 + 324 = [(n-3)^2 + 9][(n+3)^2 + 9].$$

In view of this, the given fraction can be written as

$$\frac{(7^2+9)(13^2+9)(19^2+9)(25^2+9)\ \cdots\ (55^2+9)(61^2+9)}{(1^2+9)(7^2+9)(13^2+9)(19^2+9)\ \cdots\ (49^2+9)(55^2+9)},$$

which simplifies to $\dfrac{61^2+9}{1^2+9} = \dfrac{3730}{10} = 373.$

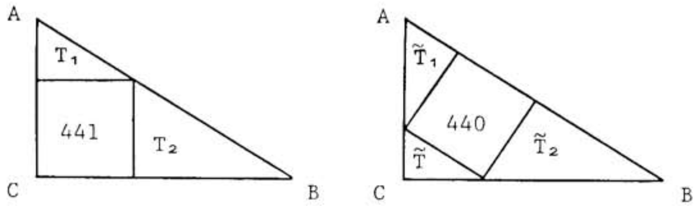
15. (462)

Let $a = BC$, $b = AC$. We will first find the hypotenuse $c$ of $\Delta ABC$ and the altitude $h$ on $c$, because these are relatively easy to compute, and because from

$$c^2 = a^2 + b^2 \quad \text{and} \quad ch = ab$$

it is easy to find $a + b$ (without first finding $a$ and $b$, which is harder!).

Consider the ratios of the areas of the smaller triangles surrounding $S_1$ and $S_2$, using the fact that all five of these triangles are similar to one another and to $\Delta ABC$. To simplify the notation, let $T$ denote both $\Delta ABC$ and its area, and similarly, let $T_1$, $T_2$, $\tilde{T}_1$, $\tilde{T}_2$ and $\tilde{T}$ denote both the triangles indicated in the figures below and their respective areas.




Since $\dfrac{\tilde{T}_1}{T_1} = \dfrac{\tilde{T}_2}{T_2} = \dfrac{440}{441}$, we find that

$$T = \tilde{T}_1 + \tilde{T}_2 + 440 + \tilde{T} = \frac{440}{441}(T_1 + T_2 + 441) + \tilde{T} = \frac{440}{441}T + \tilde{T}.$$

Therefore, $\tilde{T} = \dfrac{1}{441}T$, and hence the corresponding parts of triangles $\tilde{T}$ and $T$ are in a linear ratio of 1 to 21.

It follows that $c = 21\sqrt{440}$, since the hypotenuse of $\tilde{T}$ is $\sqrt{440}$. Moreover, $h = 21\tilde{h}$, where $\tilde{h}$ denotes the altitude of $\tilde{T}$ on its hypotenuse. Combining the latter equation with the observation $h = \tilde{h} + \sqrt{440}$, we find that $h = \dfrac{21}{20}\sqrt{440}$,

$$ab = ch = (21\sqrt{440})\left(\frac{21}{20}\sqrt{440}\right) = 21^2 \cdot 22,$$

$$(a+b)^2 = c^2 + 2ab = 21^2 \cdot 440 + 2 \cdot 21^2 \cdot 22 = 21^2 \cdot 22^2,$$

and

$$AC + CB = a + b = 21 \cdot 22 = 462.$$

Notes. More generally, if $\text{area}(S_1) = p^2$ and $\text{area}(S_2) = q^2$, one can show that

(1) $$p = ab/(a+b) \quad \text{and} \quad q = ab\sqrt{a^2+b^2}/(a^2+ab+b^2).$$

Then, if $p$ and $q$ are given, either by solving the equations in (1) simultaneously (in the unknowns $a+b$ and $ab$) or otherwise, one can show that

(2) $$a+b = p+(p^2/\sqrt{p^2-q^2}) \quad \text{and} \quad ab = p(a+b).$$

Knowing the values of $a+b$ and $ab$, one can also determine the values of $a$ and $b$ explicitly with the help of the Quadratic Formula. In the present problem they turn out to be $21(11 \pm 3\sqrt{11})$. Since these two numbers are in the ratio of $10+3\sqrt{11}$ to $1$, the accompanying figures are obviously not drawn to scale.