AIME SOLUTIONS PAMPHLET

FOR STUDENTS AND TEACHERS

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This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs. geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

It is hoped that teachers will find the time to share these solutions with their students, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution.

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1. (Answer: 60)
   Converting each of the given logarithms into exponential form gives $x^{24} = w$, $y^{40} = w$, $(xyz)^{12} = w$. It follows that

$$z^{12} = \frac{w}{x^{12}y^{12}} = \frac{w}{w^{1/2}\,w^{3/10}} = w^{1/5}.$$

   Thus $w = z^{60}$ and $\log_z w = 60$.

2. (15)
   Since $0 < p \le x \le 15$, then $|x - p| = x - p$, $|x - 15| = 15 - x$, and $|x - (p + 15)| = p + 15 - x$. Thus

$$f(x) = (x - p) + (15 - x) + (p + 15 - x) = 30 - x.$$

   It follows that $f(x)$ is least when x is greatest, and that the answer is 15.

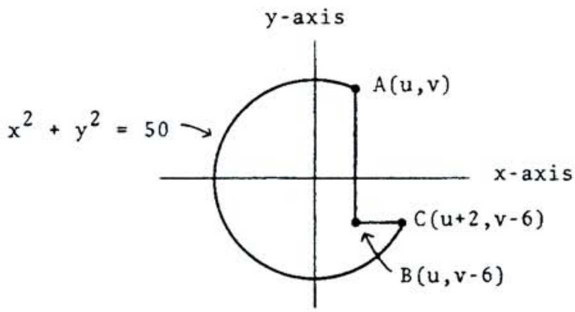
3. (20)
   Substitute to simplify. Several choices work well; the following substitution, which eliminates the radical immediately, is perhaps best. Define u to be the nonnegative number such that $u^2 = x^2 + 18x + 45$. (There is such a u, for if $x^2 + 18x + 45$ were negative, the right-hand side of the original equation would be undefined.) So

$$u^2 - 15 = 2\sqrt{u^2} = 2u \quad (\text{since } u \ge 0),$$
$$u^2 - 2u - 15 = 0,$$
$$(u - 5)(u + 3) = 0.$$

   Since $u \ge 0$, we have $u = 5$. That is, x is a solution of the original equation iff (if and only if) $x^2 + 18x + 45 = 5^2$, i.e., iff $x^2 + 18x + 20 = 0$. Both solutions to this last equation are real (why?) and their product is the constant term, 20. (Note: by being careful about the sign of u and by using iff-arguments we have avoided introducing any extraneous roots for x.)

4. (26)

We introduce a coordinate system as shown in the figure: WLOG (Without Loss Of Generality) we have put the top corner of the notch in the First Quadrant and have positioned the sides of the notch parallel to the axes. Letting $A = (u,v)$ it follows, using the given dimensions, that $B = (u,v-6)$ and $C = (u+2,v-6)$. Our problem is to determine $u^2 + (v-6)^2$. Since A and C are



both on the circle $x^2 + y^2 = 50$, we may substitute their coordinates into this equation to get the system

$$u^2 + v^2 = 50,$$
$$(u + 2)^2 + (v - 6)^2 = 50.$$

Subtracting the first equation from the second leads to $u - 3v + 10 = 0$. Solving for u and substituting into the first equation gives

$$(3v - 10)^2 + v^2 = 50,$$
$$v^2 - 6v + 5 = 0,$$
$$v = 1,\ u = -7 \quad \text{or} \quad v = 5,\ u = 5.$$

Since $A = (u,v)$ is in the First Quadrant, we have $A = (5,5)$ and $B = (5,-1)$. Finally, the square of the distance from B to the origin is $25 + 1 = 26$.

5. (4)

We are given that $x^2 + y^2 = 7$ and $x^3 + y^3 = 10$. Because we are asked to find the sum $x + y$, rather than x or y individually, we are moved to rewrite these equations so as to exhibit the sum:

$$(1) \qquad \begin{aligned} x^2 + y^2 &= (x+y)^2 - 2xy = 7, \\ x^3 + y^3 &= (x+y)^3 - 3xy(x+y) = 10. \end{aligned}$$

This further suggests defining

$$(2) \qquad s = x + y, \quad p = xy.$$

Substituting (2) into (1) gives

$$(3) \qquad \begin{aligned} s^2 - 2p &= 7, \\ s^3 - 3ps &= 10. \end{aligned}$$

Solving for p in the first line of (3) and substituting the result into the second line gives

$$(4) \quad s^3 - 3s\left(\frac{s^2-7}{2}\right) = 10 \quad \Longleftrightarrow \quad s^3 - 21s + 20 = 0.$$

An obvious root of this is $s = 1$. Dividing the cubic by $s - 1$ gives the factorization $(s-1)(s^2 + s - 20) = 0$ and thence

$$(s-1)(s-4)(s+5) = 0.$$

Thus <u>all</u> values of $s = x + y$ are real and the largest is 4.

There is a gap in this argument. Clearly any solution to (1) has led by substitution to some solution to (4), but we must also show that the largest solution for s in (4) is one which arises this way. We show more, that every solution for s in (4) arises this way. This converse is hardly automatic, as the equations are not linear.

Given any s satisfying (4), set $p = \frac{s^2-7}{2}$. Then (4) $\Longrightarrow$ (3). Next, for this pair (s,p), there necessarily exists a pair (x,y) satisfying (2), namely, the (possibly complex) roots of $z^2 - sz + p = 0$. Finally, substituting (2) into (3) gets us back to (1). That is, there is a solution (x,y) to (1) with $x + y = s$.

Alternate Solution. We use the method of Newton sums for determining

$$S_n = x^n + y^n, \quad n \geq 0,$$

where x and y are any two complex numbers. Setting $s = x + y$, $p = xy$, we have again that x and y are the roots of $z^2 - sz + p = 0$. We claim that

$$(1) \qquad S_{n+2} - s\,S_{n+1} + p\,S_n = 0, \quad n \geq 0.$$

This is shown by adding $x^n(x^2 - sx + p) = x^n \cdot 0 = 0$ and $y^n(y^2 - sy + p) = 0$. Now let x and y be the desired solutions to

$$(2) \qquad x^2 + y^2 = 7, \quad x^3 + y^3 = 10.$$

That is, we are given $S_2 = 7$ and $S_3 = 10$. We seek $S_1 = s$. We also know that $S_0 = x^0 + y^0 = 2$. Thus setting $n = 0$ and then $n = 1$ in (1), we obtain

$$(3) \qquad \begin{aligned} 7 - s^2 + 2p &= 0, \\ 10 - 7s + ps &= 0. \end{aligned}$$

Thus $p = \dfrac{s^2 - 7}{2}$ and

$$(4) \quad 10 - 7s + \frac{s^3 - 7s}{2} = 0 \iff s^3 - 21s + 20 = 0.$$

Thus, as before, $s = -5, 1, 4$ and the largest value is 4.

Also as before, the argument is not complete: every solution to (2) yields a solution to (4), but we also want the converse. The converse can be proved by methods similar to those in the first solution but a little more involved. Can the reader do it?

6. (35)

For n odd we may write

$$\begin{aligned} a_n &= (7-1)^n + (7+1)^n \\ &= \left(7^n - \tbinom{n}{1} 7^{n-1} + \cdots - 1\right) + \left(7^n + \tbinom{n}{1} 7^{n-1} + \cdots + 1\right) \\ &= 2\left(7^n + \tbinom{n}{2} 7^{n-2} + \cdots + \tbinom{n}{n-3} 7^3 + \tbinom{n}{n-1} 7\right) \end{aligned}$$

$$= 2\cdot 49\left[7^{n-2} + \binom{n}{2}7^{n-4} + \cdots + \binom{n}{n-3}7\right] + 14n.$$

It follows that $a_{83} = 49k + 14\cdot 83 = 49k + 1162$, where k is an integer. Thus, the remainder on dividing a_{83} by 49 is the same as the remainder on dividing 1162 by 49. That remainder is 35.

7. (57)

It is perhaps easier to think in terms of n knights, where $n > 4$. We observe first that there are $\binom{n}{3}$ ways of selecting 3 knights if there are no restrictions. But how many of these threesomes include at least two table neighbors?

First, there are n ways to pick three neighboring knights. (Consider each knight along with the two knights to his immediate right.) Second, there are n ways to pick two neighboring knights (as with three) followed by $(n-4)$ ways of picking a third non-neighboring knight. (We must avoid the pair and the two knights on either side.) Thus, there are $n(n-4)$ threesomes that include exactly two neighbors.

Letting P_n be the probability that at least two of the three chosen knights had been neighbors, it follows that

$$P_n = \frac{n + n(n-4)}{\binom{n}{3}} = \frac{6(n-3)}{(n-1)(n-2)} .$$

Then $P = P_{25} = \frac{11}{46}$ and the required sum is 57.

8. (61)

Let p be a prime less than 100. Then

$$n = \frac{1\cdot 2\cdot 3\cdots p\cdots 2p\cdots 3p\cdots\cdots kp\cdots 200}{(1\cdot 2\cdot 3\cdots p\cdots 2p\cdots\cdots jp\cdots 100)^2} .$$

Thus, if in addition $p^2 > 200$, we have

$$n = (\text{a } p\text{-free integer})\cdot\frac{p^k}{p^{2j}} .$$

Now, such a prime divides n iff $k > 2j$. Once we show that there is at least one prime meeting these conditions, our answer will be the largest such prime, because any p with $p^2 \le 200$ will be smaller. Since $kp \le 200$, the requirement that p be as large as possible leads to our choosing k as small as possible. Thus we take $k = 3$ and $j = 1$, for which the largest p is 61. Since $61^2 > 200$, the second display above is correct for $p = 61$. Thus 61 divides n and is the largest 2-digit prime to do so.

9. (12)

Dividing out gives

$$f(x) = 9x\sin x + \frac{4}{x\sin x} .$$

Call the first term on the right u, the second v, and note that uv is constant. This suggests using the geometric-arithmetic mean inequality:

$$\frac{u+v}{2} \ge \sqrt{uv} ,$$

where u, v are any nonnegative numbers and equality holds iff $u = v$. Applying this inequality to our u and v gives

$$\frac{f(x)}{2} \ge \sqrt{9\cdot 4} , \quad \text{or} \quad f(x) \ge 12.$$

The value 12 is actually attained iff there is an x for which

$$9x\sin x = \frac{4}{x\sin x}, \quad \text{i.e.,} \quad x^2\sin^2 x = \frac{4}{9}.$$

Since $x^2\sin^2 x$ is 0 when $x = 0$ and it exceeds 1 when $x = \pi/2$, it follows that it equals the intermediate value 4/9 somewhere between 0 and $\pi/2$. Thus the minimum value of f(x) is indeed 12.

10. (432)

We split the set of numbers of the type described into two subsets and make two separate subcounts:

1) Those numbers having two 1's. The second may be placed in any one of three positions and the two other numbers (distinct) may be placed in $9\cdot 8$ ways. Thus, there are $3\cdot 9\cdot 8 = 216$ such numbers.
2) Those numbers having two of some digit other than 1. The pair of identical digits may be selected in 9 ways and then

placed in 3 ways (positions 2 and 3, 2 and 4, or 3 and 4). The remaining number may be selected in 8 ways. Thus there are $9\cdot3\cdot8 = 216$ such numbers.

Thus, the answer is $216 + 216 = 432$. (It is coincidental, perhaps remarkable, that the two subtotals are the same. This is not so for, say, the 5-digit version of this problem. Try it!)

Alternate Solution. Consider 4 cells (for the 4-digit number) as below:

Select two of the cells as the ones that are to receive the equal digits. Call the left of these cells L and the right one R. This can be done in $\binom{4}{2} = 6$ ways. For instance, one might have

	L	R	

Starting with the first cell, fill in all cells except R with distinct numbers. This can be done in $1\cdot9\cdot8$ ways (since there is only one choice, the number 1, for the first cell, which is never R). Finally, fill R with the same number as was inserted in L. This can be done in just 1 way. Thus, the "product of ways" is $6\cdot1\cdot9\cdot8\cdot1 = 432$.

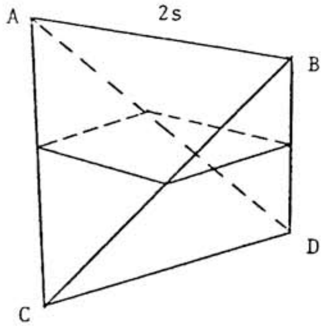
11. (288)

Consider the regular tetrahedron ABCD, shown on the next page, that has edge length 2s. Connect the midpoints of AC, BC, AD and BD. The quadrilateral thus determined is a square (why?) and the plane of this square divides the tetrahedron into two solids that are identical with the solid given in the problem. Thus, we have only to find the volume of the tetrahedron and then divide by 2.

The formula for the volume of a regular tetrahedron of side length e — its derivation requires only right-triangle trigonometry — is

$$V = \frac{\sqrt{2}}{12} e^3 .$$

Thus, substituting $e = 2s = 12\sqrt{2}$ and taking $\frac{1}{2}V$ gives 288.

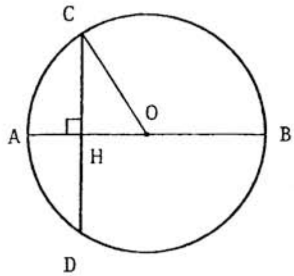


12. (65)

Let $AB = 10t + u$, where t and u are digits. Then $CD = 10u + t$. We must now find t and u for which OH will be a positive rational. Evidently, $t \geq u$; to insure that OH is positive we take $t > u$.

Since ΔOCH is a right triangle, we may use the Pythagorean Theorem to express OH in terms of AB and CD:

$$\begin{aligned} OH &= \sqrt{(OC)^2 - (CH)^2} \\ &= \frac{1}{2}\sqrt{(10t+u)^2 - (10u+t)^2} \\ &= \frac{3}{2}\sqrt{11(t^2-u^2)} . \end{aligned}$$



It follows that OH is rational iff $\sqrt{11(t^2-u^2)}$ is rational. But the square root of an integer is rational only if it is integral. Thus, we must find t and u for which $11(t^2-u^2)$ is a perfect square, and this will be the case only if there is a positive integer m such that

$$t^2 - u^2 = 11m^2 ,$$

$$(t-u)(t+u) = 11m^2 .$$

Now 11 cannot divide $t-u$. (Why?) Therefore, it must divide $t+u$. But this is possible only if $t+u$ is 11. (Why?) It follows that $t-u = m^2$. Thus, we seek two numbers whose sum is 11 and whose difference is a perfect square.

Finally, we examine all t and u $(t > u)$ for which $t+u = 11$: $t=9$, $u=2$; $t=8$, $u=3$; $t=7$, $u=4$; and $t=6$, $u=5$. Only in the last case is $t-u$ a perfect square. Thus AB = 65.

Note. Without the condition that OH is positive rational, the problem has many answers. For example, take AB = 52 and CD = 25 (OH is irrational), or AB = 77 = CD (OH is zero). It happens that (in base ten) there is just one case in which OH is a positive rational.

13. (448)

It is easier, perhaps, to generalize the problem (ever so slightly) by considering the alternating sums for all subsets of $\{1, 2, 3, \ldots, n\}$, that is, by including the empty set. To include the empty set without affecting the answer we have only to declare that its alternating sum be 0. The subsets of $\{1, 2, 3, \ldots, n\}$ may be divided into two kinds: those that do not contain n and those that do. Moreover, each subset of the first kind may be paired — in a one-to-one correspondence — with a subset of the second kind as follows:

$$\{a_1, a_2, a_3, \ldots, a_i\} \longleftrightarrow \{n, a_1, a_2, a_3, \ldots, a_i\} .$$

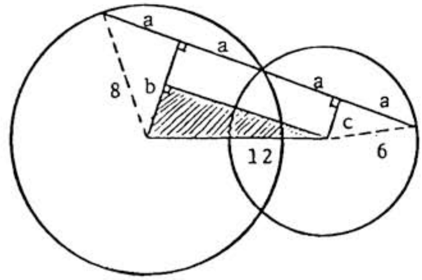
(For the empty set we have the correspondence $\phi \longleftrightarrow \{n\}$.) Then, assuming $n > a_1 > a_2 > \ldots > a_i$, the sum of the alternating sums for each such pair of subsets is given by

$$(a_1 - a_2 + a_3 - \cdots \pm a_i) + (n - a_1 + a_2 - a_3 + \cdots \mp a_i) = n.$$

And since there are 2^{n-1} such pairs of subsets (why?), the required sum is $n\,2^{n-1}$. Finally, taking $n = 7$. we obtain 448.

14. (130)

We complete the figure as shown. Applying the Pythagorean Theorem to the shaded triangle gives



$$(2a)^2 + (b-c)^2 = 12^2.$$

But $b = \sqrt{64-a^2}$ and $c = \sqrt{36-a^2}$. Thus

$$4a^2 + \left(\sqrt{64-a^2} - \sqrt{36-a^2}\right)^2 = 144.$$

Simplifying gives $a^2 - 22 = \sqrt{(64-a^2)(36-a^2)}$. Squaring and simplifying gives $4a^2 = 130$. Thus $(2a)^2 = 4a^2 = 130$.

15. (175)

Consider the family of all chords emanating from A. Then the locus of the endpoints of these chords is the given circle of radius 5 and the locus of their midpoints is an internally tangent circle (at A) of radius 5/2. (We have simply shrunk the given circle relative to A by a factor of 1/2.) Now BC must be tangent to the smaller circle. For if it cut that circle twice, as in Figure 1, then there would be two chords emanating from A that are bisected by BC.

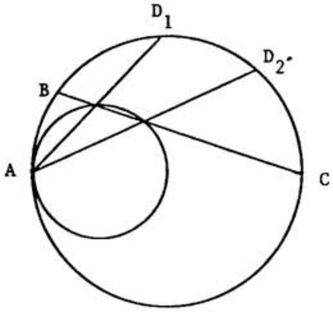


Figure 1

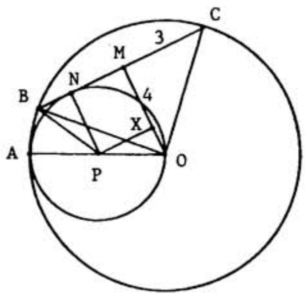


Figure 2

Next consider Figure 2, where O and P are the centers of the two circles and BC is tangent to the smaller circle (at N) as described before. We introduce the following construction lines: OA (through P), OB, OC, $MO \perp BC$, $NP \perp BC$, and $PX \| BC$. We seek $\sin \angle BOA$. We will find this by first finding all three sides of ΔBOP. We have

1. $MC = 3$ and $MO = \sqrt{5^2 - 3^2} = 4$.
2. $NP = \frac{1}{2} AO = \frac{5}{2}$.
3. MNPX is a rectangle, so $MX = \frac{5}{2}$ and $OX = \frac{3}{2}$.
4. ΔPXO is a right triangle with $OP = \frac{5}{2}$ and $OX = \frac{3}{2}$, so $PX = \frac{4}{2} = 2$.
5. $MN = 2$ so $BN = 1$.
6. ΔBNP is a right triangle, so $BP^2 = 1 + \frac{25}{4} = \frac{29}{4}$.

We now have all three sides of ΔBOP, so by the Law of Cosines applied to $\theta = \angle BOP$ we obtain

$$\frac{29}{4} = 25 + \frac{25}{4} - 2\cdot 5\cdot \frac{5}{2}\cos\theta,$$

$$\cos\theta = \frac{24}{25}.$$

Thus $\sin\theta = \sqrt{1 - \left(\frac{24}{25}\right)^2} = \frac{7}{25}$ and the desired answer is 175.