AMERICAN MATHEMATICS COMPETITIONS

7th ANNUAL
AMERICAN INVITATIONAL MATHEMATICS EXAMINATION
(AIME)
TUESDAY, MARCH 21, 1989

Sponsored by

Mathematical Association of America
Society of Actuaries   Mu Alpha Theta
National Council of Teachers of Mathematics
Casualty Actuarial Society   American Statistical Association
American Mathematical Association of Two-Year Colleges
American Mathematical Society

1. DO NOT OPEN THIS BOOKLET UNTIL TOLD TO DO SO BY YOUR PROCTOR.
2. This is a 15-question, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers; i.e., there is neither partial credit nor a penalty for wrong answers.
3. Scratch paper, graph paper, ruler, compass, protractor and eraser are permitted. *Calculators and slide rules are not permitted.*
4. Please print the following:

Name: ______________________________
Last   First   Middle initial

Home address: ______________________________
Street Address

____
City   State or Province   Zip or Postcode

____
Home Phone including Area Code   Sex (M or F)   Your age

____
Full Name of School   Grade Level (e.g.,11)

5. My score on the 1989 AHSME was ☐   I took the 1989 AHSME on (date): ____________
6. A combination of the AIME and AHSME scores is used to determine eligibility for participation in the U. S. A. Mathematical Olympiad (USAMO) to be given on April 25, 1989. Please check one box:

   If I qualify for the USAMO, I agree to take it.   YES ☐   NO ☐

   (*Your school must also agree to administer the USAMO before you can take it.*)
7. Record all your answers, and certain other information, on a computer card. Your Examination Manager will instruct you how to complete the card. Only the computer card and this cover will be collected from you.

1. Compute $\sqrt{(31)(30)(29)(28)+1}$.

2. Ten points are marked on a circle. How many distinct convex polygons of three or more sides can be drawn using some (or all) of the ten points as vertices? (Polygons are distinct unless they have exactly the same vertices.)

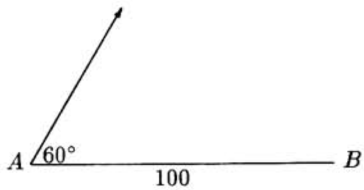
3. Suppose n is a positive integer and d is a single digit in base 10. Find n if

$$\frac{n}{810} = 0.\text{d}25\text{d}25\text{d}25\ldots.$$

4. If $a < b < c < d < e$ are consecutive positive integers such that $b + c + d$ is a perfect square and $a + b + c + d + e$ is a perfect cube, what is the smallest possible value of c?

5. When a certain biased coin is flipped 5 times, the probability of getting heads exactly once is not equal to 0 and is the same as that of getting heads exactly twice. Let i/j, in lowest terms, be the probability that the coin comes up heads exactly 3 times out of 5. Find $i + j$.

6. Two skaters, Allie and Billie, are at points A and B, respectively, on a flat, frozen lake. The distance between A and B is 100 meters. Allie leaves A and skates at a speed of 8 meters per second along a straight line that makes an angle of 60° with $\overline{AB}$, as shown. At the same time that Allie leaves A, Billie leaves B at a speed of 7 meters per second and follows the straight line path that produces the earliest possible meeting of the two skaters, given their speeds. How many meters does Allie skate before meeting Billie?



7. If the integer k is added to each of the numbers 36, 300 and 596, one obtains the squares of three consecutive terms of an arithmetic sequence. Find k.

8. Assume that $x_1, x_2, \ldots, x_7$ are real numbers such that

$$\begin{aligned} x_1 + 4x_2 + 9x_3 + 16x_4 + 25x_5 + 36x_6 + 49x_7 &= 1 \\ 4x_1 + 9x_2 + 16x_3 + 25x_4 + 36x_5 + 49x_6 + 64x_7 &= 12 \\ 9x_1 + 16x_2 + 25x_3 + 36x_4 + 49x_5 + 64x_6 + 81x_7 &= 123. \end{aligned}$$

Find the value of

$$16x_1 + 25x_2 + 36x_3 + 49x_4 + 64x_5 + 81x_6 + 100x_7.$$

9. One of Euler's conjectures was disproved in the 1960s by three American mathematicians when they showed that there is a positive integer n such that

$$133^5 + 110^5 + 84^5 + 27^5 = n^5.$$

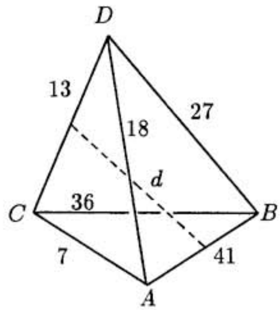
Find the value of n.

10. Let a, b, c be the three sides of a triangle, and let α, β, γ, respectively, be the angles opposite them. If $a^2 + b^2 = 1989c^2$, find

$$\frac{\cot \gamma}{\cot \alpha + \cot \beta}.$$

11. A sample of 121 integers is given, each between 1 and 1000 inclusive, with repetitions allowed. The sample has a unique mode (most frequent value). Let D be the difference between the mode and the arithmetic mean of the sample. If D is as large as possible, what is $\lfloor D \rfloor$? (For real x, $\lfloor x \rfloor$ is the greatest integer less than or equal to x.)

12. Let $ABCD$ be a tetrahedron with $AB = 41$, $AC = 7$, $AD = 18$, $BC = 36$, $BD = 27$, and $CD = 13$, as shown in the figure. Let d be the distance between the midpoints of edges $\overline{AB}$ and $\overline{CD}$. Find d^2.



13. Let S be a subset of $\{1, 2, 3, \ldots, 1989\}$ such that no two members of S differ by 4 or 7. What is the largest number of elements S can have?

14. Given a positive integer n, it can be shown that every complex number of the form $r + si$, where r and s are integers, can be uniquely expressed in the base $-n + i$ using the integers $0, 1, 2, \ldots, n^2$ as "digits." That is, the equation

$$r + si = a_m(-n+i)^m + a_{m-1}(-n+i)^{m-1} + \cdots + a_1(-n+i) + a_0$$

is true for a unique choice of non-negative integer m and digits $a_0, a_1, \ldots, a_m$ chosen from the set $\{0, 1, 2, \ldots, n^2\}$, with $a_m \neq 0$. We then write

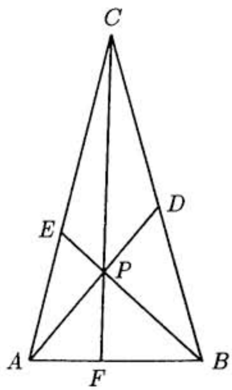
$$r + si = (a_m a_{m-1} \ldots a_1 a_0)_{-n+i}$$

to denote the base $-n + i$ expansion of $r + si$. There are only finitely many integers $k + 0i$ that have four-digit expansions

$$k = (a_3 a_2 a_1 a_0)_{-3+i} \qquad a_3 \neq 0.$$

Find the sum of all such k.

15. Point P is inside $\triangle ABC$. Line segments $\overline{APD}$, $\overline{BPE}$ and $\overline{CPF}$ are drawn with D on $\overline{BC}$, E on $\overline{CA}$, and F on $\overline{AB}$ (see the figure at the right). Given that $AP = 6$, $BP = 9$, $PD = 6$, $PE = 3$ and $CF = 20$, find the area of $\triangle ABC$.



SOLUTIONS

A 1989 Solutions Pamphlet will be sent to exam managers within a few weeks.

WRITE TO US!

Questions and comments about the problems and solutions for this AIME (but **not** requests for the Solutions Pamphlet) should be addressed to:

Prof Elgin H Johnston, AIME Chairman
Department of Mathematics
Iowa State University, Ames, IA 50011 USA

Comments about administrative arrangements and orders for any publications listed below should be addressed to:

Prof Walter E Mientka, CAMC Executive Director
Department of Mathematics and Statistics
University of Nebraska, Lincoln, NE 68588-0322 USA

1989 USAMO

The USA Mathematical Olympiad is a 5-question, $3\frac{1}{2}$ hour, essay-type examination. Top-scoring AHSME/AIME students will be invited to take the USAMO on April 25, 1989. (See the AHSME or AIME Teachers' Manual for more details.) The best way to prepare for the USAMO is to study the exams from previous years and to review the contents of the ARBELOS. Copies may be ordered as indicated below.

PUBLICATIONS

MINIMUM ORDER: $5 (before handling fee), US FUNDS ONLY. Canada and US orders must be prepaid. Orders are mailed 4th class, unless you specify 1st class, in which case add $3.00 or 20% of total order, whichever is larger. Make checks payable to the American Mathematics Competitions (AMC).

FOREIGN ORDERS: do **NOT** prepay; an invoice will be sent.

Examinations: Each price is for **one** copy of an exam **and** its solutions for one year. Specify the years you want and how many copies of each. All prices effective to July 1, 1989.

- **AJHSME** (Junior High Exam), 1985-1988, 50 cents per copy per year.
- **AHSME** 1972-89, 50 cents per copy per year.
- **AIME** 1983-88, $1 per copy per year.
- USA and International Mathematical **Olympiads** (together), 1976-88, $1 per copy per year.
- National **Summary** of Results and Awards, 1976-88, $4 per copy per year.

Books (Exams and solutions):

- Contest Problem Book I, AHSMEs 1950-60, $8.50.
- Contest Problem Book II, AHSMEs 1961-65, $8.50.
- Contest Problem Book III, AHSMEs 1966-72, $9.50.
- Contest Problem Book IV, AHSMEs 1973-82, $10.50.
- International Mathematical Olympiads, 1959-77, $9.50.
- International Mathematical Olympiads, 1978-85, $11.00.

Journal

- The ARBELOS (short articles and challenging problems); recommended especially for AIME and USAMO qualifiers. Six volumes, 1982-1987, $6.00 each. Canada and APO/FPO—add $3 per volume.