AMERICAN MATHEMATICS COMPETITIONS

# 6th ANNUAL
# AMERICAN INVITATIONAL MATHEMATICS EXAMINATION
# (AIME)

## TUESDAY, MARCH 22, 1988

Sponsors:
MATHEMATICAL ASSOCIATION OF AMERICA
SOCIETY OF ACTUARIES MU ALPHA THETA
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CASUALTY ACTUARIAL SOCIETY
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INSTRUCTIONS

1. *Do not open* this book until told to do so.
2. This is a 15-question, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers; i.e., neither partial credit nor a penalty for wrong answers will be given.
3. All your answers, and certain other information, are to be recorded on a computer card. Your Examination Manager will instruct you how to fill out the card after you have finished with these instructions. Only the computer card and this cover sheet will be collected from you.
4. Scratch paper, graph paper, ruler, compass and eraser are permitted. *Calculators and slide rules are not permitted.*
5. Please print the following:

Last Name _____ First Name _____ Middle Initial _____

Home Address _____

City _____ State or Province _____ Zip or Postcode _____

Home Phone including Area Code _____ Sex (M or F) _____ Your age _____

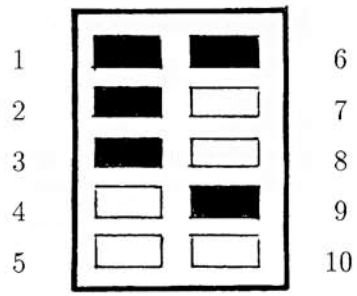
Full Name of School _____ Grade Level (e.g., 11) _____

6. My score on the 1988 AHSME was ☐
   I took the 1988 AHSME on _____ (date)

7. This AIME is the qualifying examination for the U.S.A. Mathematical Olympiad (USAMO) to be given on April 26, 1988. Please check one box:
   If I qualify for the USAMO, I agree to take it. YES ☐ NO ☐

   Your school must also agree to administer the USAMO before you can take it.

. One commercially available ten-button lock may be opened by depressing — in any order — the correct five buttons. The sample shown at right has $\{1,2,3,6,9\}$ as its combination. Suppose that these locks are redesigned so that sets of as many as nine buttons or as few as one button could serve as combinations. How many additional combinations would this allow?




. For any positive integer $k$, let $f_1(k)$ denote the square of the sum of the digits of $k$. For $n \geq 2$, let $f_n(k) = f_1(f_{n-1}(k))$. Find $f_{1988}(11)$.

. Find $(\log_2 x)^2$ if $\log_2(\log_8 x) = \log_8(\log_2 x)$.

. Suppose that $|x_i| < 1$ for $i = 1, 2, \ldots, n$. Suppose further that

$$|x_1| + |x_2| + \ldots + |x_n| = 19 + |x_1 + x_2 + \ldots + x_n|.$$

What is the smallest possible value of $n$?

. Let $m/n$, in lowest terms, be the probability that a randomly chosen positive divisor of $10^{99}$ is an integer multiple of $10^{88}$. Find $m + n$.

. It is possible to place positive integers into the twenty-one vacant squares of the $5 \times 5$ square show on the right so that the numbers in each row and column form arithmetic sequences. Find the number that must occupy the vacant square marked by the asterisk $(*)$.

| | | | | |
|---|---|---|---|---|
| | | | * | |
| | 74 | | | |
| | | | | 186 |
| | | 103 | | |
| 0 | | | | |

7. In $\triangle ABC, \tan(\angle CAB) = 22/7$ and the altitude from $A$ divides $BC$ into segments of length 3 and 17. What is the area of $\triangle ABC$?

8. The function $f$, defined on the set of ordered pairs of positive integers, satisfies the following properties:

$$f(x,x) = x, \quad f(x,y) = f(y,x), \quad \text{and} \quad (x+y)f(x,y) = yf(x,x+y).$$

Calculate $f(14,52)$.

9. Find the smallest positive integer whose cube ends in 888.

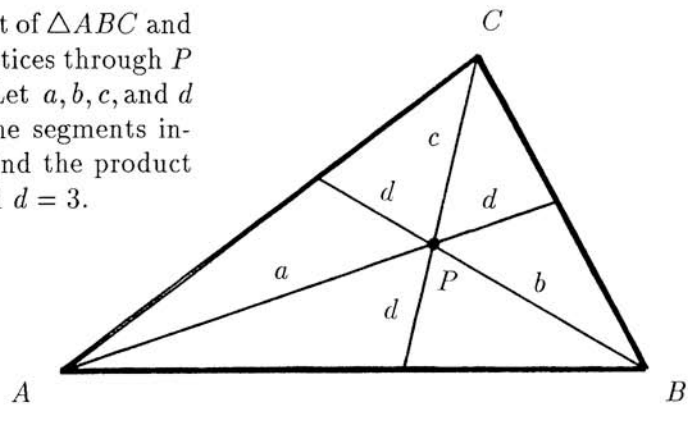
10. A convex polyhedron has for its faces 12 squares, 8 regular hexagons, and 6 regular octagons. At each vertex of the polyhedron one square, one hexagon, and one octagon meet. How many segments joining vertices of the polyhedron lie in the interior of the polyhedron rather than along an edge or a face?

11. Let $w_1, w_2, \ldots, w_n$ be complex numbers. A line $L$ in the complex plane is called a *mean line* for the points $w_1, w_2, \ldots, w_n$ if $L$ contains points (complex numbers) $z_1, z_2, \ldots, z_n$ such that

$$\sum_{k=1}^{n}(z_k - w_k) = 0.$$

For the numbers $w_1 = 32 + 170i$, $w_2 = -7 + 64i$, $w_3 = -9 + 200i$, $w_4 = 1 + 27i$, and $w_5 = -14 + 43i$ there is a unique mean line with $y-$intercept 3. Find the slope of this mean line.

12. Let $P$ be an interior point of $\triangle ABC$ and extend lines from the vertices through $P$ to the opposite sides. Let $a, b, c,$ and $d$ denote the lengths of the segments indicated in the figure. Find the product $abc$ if $a + b + c = 43$ and $d = 3$.



3. Find $a$ if $a$ and $b$ are integers such that $x^2 - x - 1$ is a factor of $ax^{17} + bx^{16} + 1$.

4. Let $C$ be the graph of $xy = 1$, and denote by $C^*$ the reflection of $C$ in the line $y = 2x$. Let the equation of $C^*$ be written in the form

$$12x^2 + bxy + cy^2 + d = 0.$$

Find the product $bc$.

5. In an office, at various times during the day, the boss gives the secretary a letter to type, each time putting the letter on top of the pile in the secretary's in-box. When there is time, the secretary takes the top letter off the pile and types it. There are nine letters to be typed during the day, and the boss delivers them in the order $1, 2, 3, 4, 5, 6, 7, 8, 9$.

While leaving for lunch, the secretary tells a colleague that letter 8 has already been typed, but says nothing else about the morning's typing. The colleague wonders which of the nine letters remain to be typed after lunch and in what order they will be typed. Based upon the above information, how many such *after-lunch typing orders* are possible? (That there are no letters left to be typed is one of the possibilities.)

## SOLUTIONS

A 1988 Solutions Pamphlet will be sent to you for a nominal fee; for details, see below.

## WRITE TO US!

Address questions and comments about the problems & solutions for this AIME to:

Professor Elgin H. Johnston
Department of Mathematics
Iowa State University, Ames, IA 50011 USA

Comments about administrative arrangements, and orders for any of the publications listed below, should be addressed to:

Professor Walter E. Mientka, CAMC Executive Director
Department of Mathematics & Statistics
University of Nebraska, Lincoln, NE 68588-0322 USA

## 1988 USAMO

The USA Mathematical Olympiad is a 5-question, 3½-hour essay-type examination. Top scoring AHSME/AIME students will be invited to take the USAMO on April 26, 1988, see the AHSME or AIME Teachers' Manual for more details. The best way to prepare for the USAMO is to study the exams from previous years and to review the contents of the ARBELOS. The procedure to be used to purchase these publications is indicated below.

## PUBLICATIONS

MINIMUM ORDER: $5 (before handling fee), **US FUNDS ONLY.** Individual orders from US and Canada must be prepaid. Orders mailed 4th Class, unless you wish 1st Class, in which case add a 20% handling fee. Make checks payable to MAA/CAMC.

**FOREIGN ORDERS:** do **NOT** prepay; an invoice will be sent.

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Examinations. **Each price is for one copy of an exam and its solutions for one year.** Specify the years you want and how many copies of each. All prices effective to July 1, 1988.

AJHSME (Junior High Exam), 1985-87, 50¢ per copy.
AHSME 1972-88, 50¢ per copy per year.
AIME, 1983-88, $1 per copy per year.
USA and International Mathematical Olympiads (together), 1976-87, $1 per copy per year.
National Summary of Results and Awards, 1980-87, $3 per copy per year.

Books.

Contest Problem Book I ($8.50), AHSME exams and solutions, 1950-60.
Contest Problem Book II ($8.50), AHSME exams and solutions, 1961-65.
Contest Problem Book III ($9.50), AHSME exams and solutions, 1966-72.
Contest Problem Book IV ($10.50), AHSME exams and solutions, 1973-82.
International Mathematics Olympiads ($9.50), exams and solutions, 1959-77.
International Mathematics Olympiads ($11.00), exams and solutions, 1978-85.

Journal. The ARBELOS contains short articles and challenging problems; recommended especially for AIME and USAMO qualifiers. Back volumes (5 issues per year) from 1982-86 available at $5.50 each. Canadian and APO/FPO orders–add $3 per volume.