AMERICAN MATHEMATICS COMPETITIONS

17th ANNUAL AMERICAN INVITATIONAL MATHEMATICS EXAMINATION (AIME)

TUESDAY, MARCH 16, 1999

Sponsored by

Mathematical Association of America
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1. DO NOT OPEN THIS BOOKLET UNTIL TOLD TO DO SO BY YOUR PROCTOR.
2. This is a 15-question, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers; i.e., there is neither partial credit nor a penalty for wrong answers.
3. No aids other than scratch paper, graph paper, ruler, compass, and protractor are permitted. In particular, **calculators are <u>not</u> permitted.**
4. A combination of the AIME and AHSME scores is used to determine eligibility for participation in the U.S.A. Mathematical Olympiad (USAMO). The USAMO will be given on TUESDAY, April 27, 1999.
5. Record all your answers, and certain other information, on the AIME answer form. Only the answer form will be collected from you.

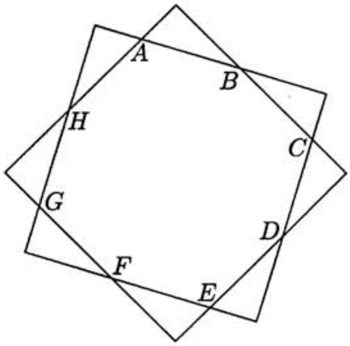
This examination was prepared during the tenure of American Mathematics Competitions Executive Director, Dr. Walter E. Mientka

1. Find the smallest prime that is the fifth term of an increasing arithmetic sequence, all four preceding terms also being prime.

2. Consider the parallelogram with vertices $(10, 45)$, $(10, 114)$, $(28, 153)$, and $(28, 84)$. A line through the origin cuts this figure into two congruent polygons. The slope of the line is m/n, where m and n are relatively prime positive integers. Find $m + n$.

3. Find the sum of all positive integers n for which $n^2 - 19n + 99$ is a perfect square.

4. The two squares shown share the same center O and have sides of length 1. The length of $\overline{AB}$ is $43/99$ and the area of octagon $ABCDEFGH$ is m/n, where m and n are relatively prime positive integers. Find $m + n$.



5. For any positive integer x, let $S(x)$ be the sum of the digits of x, and let $T(x)$ be $|S(x+2) - S(x)|$. For example, $T(199) = |S(201) - S(199)| = |3 - 19| = 16$. How many values $T(x)$ do not exceed 1999?

6. A transformation of the first quadrant of the coordinate plane maps each point (x, y) to the point $(\sqrt{x}, \sqrt{y})$. The vertices of quadrilateral $ABCD$ are $A = (900, 300)$, $B = (1800, 600)$, $C = (600, 1800)$, and $D = (300, 900)$. Let k be the area of the region enclosed by the image of quadrilateral $ABCD$. Find the greatest integer that does not exceed k.

7. There is a set of 1000 switches, each of which has four positions, called A, B, C, and D. When the position of any switch changes, it is only from A to B, from B to C, from C to D, or from D to A. Initially each switch is in position A. The switches are labeled with the 1000 different integers $2^x 3^y 5^z$, where x, y, and z take on the values $0, 1, \ldots, 9$. At step i of a 1000-step process, the ith switch is advanced one step, and so are all the other switches whose labels divide the label on the ith switch. After step 1000 has been completed, how many switches will be in position A?

8. Let $\mathcal{T}$ be the set of ordered triples (x, y, z) of nonnegative real numbers that lie in the plane $x + y + z = 1$. Let us say that (x, y, z) *supports* (a, b, c) when exactly two of the following are true: $x \geq a$, $y \geq b$, $z \geq c$. Let $\mathcal{S}$ consist of those triples in $\mathcal{T}$ that support $\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right)$. The area of $\mathcal{S}$ divided by the area of $\mathcal{T}$ is m/n, where m and n are relatively prime positive integers. Find $m + n$.

9. A function f is defined on the complex numbers by $f(z) = (a+bi)z$, where a and b are positive numbers. This function has the property that the image of each point in the complex plane is equidistant from that point and the origin. Given that $|a + bi| = 8$ and that $b^2 = m/n$, where m and n are relatively prime positive integers, find $m + n$.

10. Ten points in the plane are given, with no three collinear. Four distinct segments joining pairs of these points are chosen at random, all such segments being equally likely. The probability that some three of the segments form a triangle whose vertices are among the ten given points is m/n, where m and n are relatively prime positive integers. Find $m + n$.

11. Given that $\displaystyle\sum_{k=1}^{35} \sin 5k = \tan \frac{m}{n}$, where angles are measured in degrees, and m and n are relatively prime positive integers that satisfy $\dfrac{m}{n} < 90$, find $m + n$.

12. The inscribed circle of triangle ABC is tangent to $\overline{AB}$ at P, and its radius is 21. Given that $AP = 23$ and $PB = 27$, find the perimeter of the triangle.

13. Forty teams play a tournament in which every team plays every other team exactly once. No ties occur, and each team has a 50% chance of winning any game it plays. The probability that no two teams win the same number of games is m/n, where m and n are relatively prime positive integers. Find $\log_2 n$.

14. Point P is located inside triangle ABC so that angles PAB, PBC, and PCA are all congruent. The sides of the triangle have lengths $AB = 13$, $BC = 14$, and $CA = 15$, and the tangent of angle PAB is m/n, where m and n are relatively prime positive integers. Find $m + n$.

15. Consider the paper triangle whose vertices are (0,0), (34,0), and (16,24). The vertices of its *midpoint triangle* are the midpoints of its sides. A triangular pyramid is formed by folding the triangle along the sides of its midpoint triangle. What is the volume of this pyramid?

Your Exam Manager will be sent a copy of the 1999 AIME Solutions Pamphlet.

WRITE TO US!

Correspondence about the problems and solutions should be addressed to:

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1999 USAMO

THE USA MATHEMATICAL OLYMPIAD (USAMO) is a 6-question, 6-hour, essay-type examination. The USAMO will be held on Tuesday, April 27, 1999. Your teacher has more details on who qualifiers for the USAMO in the AHSME and AIME Teachers' Manuals. The best way to prepare for the USAMO is to study previous years of these exams, the World Olympiad Problems/Solutions and review the contents of the ARBELOS. Copies may be ordered as indicated below.

PUBLICATIONS

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Examinations: Each price is for **one** copy of an exam **and** its solutions for one year. Specify the years you want and how many copies of each. All prices effective to September 1, 1999.

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The Arbelos

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- USAMO qualifiers. Six volumes, $8.00 each.