Tuesday, FEBRUARY 6, 2007

58ᵗʰ Annual American Mathematics Contest 12

AMC 12
CONTEST A

THE MATHEMATICAL ASSOCIATION OF AMERICA
American Mathematics Competitions

1. DO NOT OPEN THIS BOOKLET UNTIL YOUR PROCTOR GIVES THE SIGNAL TO BEGIN.

2. This is a 25-question, multiple choice test. Each question is followed by answers marked A, B, C, D and E. Only one of these is correct.

3. Mark your answer to each problem on the AMC 12 Answer Form with a #2 pencil. Check the blackened circles for accuracy and erase errors and stray marks completely. Only answers properly marked on the answer form will be graded.

4. SCORING: You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.

5. No aids are permitted other than scratch paper, graph paper, ruler, compass, protractor, erasers and calculators that are accepted for use on the SAT. No problems on the test will *require* the use of a calculator.

6. Figures are not necessarily drawn to scale.

7. Before beginning the test, your proctor will ask you to record certain information on the answer form. When your proctor gives the signal, begin working the problems. You will have **75 MINUTES** to complete the test.

8. When you finish the exam, *sign your name* in the space provided on the Answer Form.

   Students who score 100 or above or finish in the top 5% on this AMC 12 will be invited to take the 25ᵗʰ annual American Invitational Mathematics Examination (AIME) on Tuesday, March 13, 2007 or Wednesday, March 28, 2007. More details about the AIME and other information are on the back page of this test booklet.

1. One ticket to a show costs \$20 at full price. Susan buys 4 tickets using a coupon that gives her a 25% discount. Pam buys 5 tickets using a coupon that gives her a 30% discount. How many more dollars does Pam pay than Susan?

   (A) 2   **(B)** 5   **(C)** 10   **(D)** 15   **(E)** 20

2. An aquarium has a rectangular base that measures 100 cm by 40 cm and has a height of 50 cm. It is filled with water to a height of 40 cm. A brick with a rectangular base that measures 40 cm by 20 cm and a height of 10 cm is placed in the aquarium. By how many centimeters does the water rise?

   (A) 0.5   **(B)** 1   **(C)** 1.5   **(D)** 2   **(E)** 2.5

3. The larger of two consecutive odd integers is three times the smaller. What is their sum?

   (A) 4   **(B)** 8   **(C)** 12   **(D)** 16   **(E)** 20

4. Kate rode her bicycle for 30 minutes at a speed of 16 mph, then walked for 90 minutes at a speed of 4 mph. What was her overall average speed in miles per hour?

   (A) 7   **(B)** 9   **(C)** 10   **(D)** 12   **(E)** 14

5. Last year Mr. John Q. Public received an inheritance. He paid 20% in federal taxes on the inheritance, and paid 10% of what he had left in state taxes. He paid a total of \$10,500 for both taxes. How many dollars was the inheritance?

   (A) 30,000   **(B)** 32,500   **(C)** 35,000   **(D)** 37,500   **(E)** 40,000

6. Triangles ABC and ADC are isosceles with $AB = BC$ and $AD = DC$. Point D is inside $\triangle ABC$, $\angle ABC = 40°$, and $\angle ADC = 140°$. What is the degree measure of $\angle BAD$?

   (A) 20   **(B)** 30   **(C)** 40   **(D)** 50   **(E)** 60

7. Let $a,b,c,d,$ and e be five consecutive terms in an arithmetic sequence, and suppose that $a + b + c + d + e = 30$. Which of the following can be found?

   (A) a **(B)** b **(C)** c **(D)** d **(E)** e

8. A star-polygon is drawn on a clock face by drawing a chord from each number to the fifth number counted clockwise from that number. That is, chords are drawn from 12 to 5, from 5 to 10, from 10 to 3, and so on, ending back at 12. What is the degree measure of the angle at each vertex in the star-polygon?

   (A) 20   **(B)** 24   **(C)** 30   **(D)** 36   **(E)** 60

9. Yan is somewhere between his home and the stadium. To get to the stadium he can walk directly to the stadium, or else he can walk home and then ride his bicycle to the stadium. He rides 7 times as fast as he walks, and both choices require the same amount of time. What is the ratio of Yan's distance from his home to his distance from the stadium?

(**A**) $\frac{2}{3}$ (**B**) $\frac{3}{4}$ (**C**) $\frac{4}{5}$ (**D**) $\frac{5}{6}$ (**E**) $\frac{6}{7}$

10. A triangle with side lengths in the ratio 3:4:5 is inscribed in a circle of radius 3. What is the area of the triangle?

(**A**) 8.64   (**B**) 12   (**C**) 5π (**D**) 17.28   (**E**) 18

11. A finite sequence of three-digit integers has the property that the tens and units digits of each term are, respectively, the hundreds and tens digits of the next term, and the tens and units digits of the last term are, respectively, the hundreds and tens digits of the first term. For example, such a sequence might begin with terms 247, 475, and 756 and end with the term 824. Let S be the sum of all the terms in the sequence. What is the largest prime number that always divides S?

(**A**) 3   (**B**) 7   (**C**) 13   (**D**) 37   (**E**) 43

12. Integers a, b, c, and d, not necessarily distinct, are chosen independently and at random from 0 to 2007, inclusive. What is the probability that $ad - bc$ is even?

(**A**) $\frac{3}{8}$ (**B**) $\frac{7}{16}$ (**C**) $\frac{1}{2}$ (**D**) $\frac{9}{16}$ (**E**) $\frac{5}{8}$

13. A piece of cheese is located at $(12, 10)$ in a coordinate plane. A mouse is at $(4, -2)$ and is running up the line $y = -5x + 18$. At the point (a, b) the mouse starts getting farther from the cheese rather than closer to it. What is $a + b$?

(**A**) 6   (**B**) 10   (**C**) 14   (**D**) 18   (**E**) 22

14. Let a, b, c, d, and e be distinct integers such that

$$(6 - a)(6 - b)(6 - c)(6 - d)(6 - e) = 45.$$

What is $a + b + c + d + e$?

(**A**) 5   (**B**) 17   (**C**) 25   (**D**) 27   (**E**) 30

15. The set $\{3, 6, 9, 10\}$ is augmented by a fifth element n, not equal to any of the other four. The median of the resulting set is equal to its mean. What is the sum of all possible values of n?

(**A**) 7   (**B**) 9   (**C**) 19   (**D**) 24   (**E**) 26

16. How many three-digit numbers are composed of three distinct digits such that one digit is the average of the other two?

   (A) 96   **(B)** 104   **(C)** 112   **(D)** 120   **(E)** 256

17. Suppose that $\sin a + \sin b = \sqrt{5/3}$ and $\cos a + \cos b = 1$. What is $\cos(a - b)$?

   (A) $\sqrt{\dfrac{5}{3}} - 1$ **(B)** $\dfrac{1}{3}$ **(C)** $\dfrac{1}{2}$ **(D)** $\dfrac{2}{3}$ **(E)** 1

18. The polynomial $f(x) = x^4 + ax^3 + bx^2 + cx + d$ has real coefficients, and $f(2i) = f(2 + i) = 0$. What is $a + b + c + d$?

   (A) 0   **(B)** 1   **(C)** 4   **(D)** 9   **(E)** 16

19. Triangles ABC and ADE have areas 2007 and 7002, respectively, with $B = (0, 0)$, $C = (223, 0)$, $D = (680, 380)$, and $E = (689, 389)$. What is the sum of all possible x-coordinates of A?

   (A) 282   **(B)** 300   **(C)** 600   **(D)** 900   **(E)** 1200

20. Corners are sliced off a unit cube so that the six faces each become regular octagons. What is the total volume of the removed tetrahedra?

   (A) $\dfrac{5\sqrt{2} - 7}{3}$ **(B)** $\dfrac{10 - 7\sqrt{2}}{3}$ **(C)** $\dfrac{3 - 2\sqrt{2}}{3}$ **(D)** $\dfrac{8\sqrt{2} - 11}{3}$

   (E) $\dfrac{6 - 4\sqrt{2}}{3}$

21. The sum of the zeros, the product of the zeros, and the sum of the coefficients of the function $f(x) = ax^2 + bx + c$ are equal. Their common value must also be which of the following?

   (A) the coefficient of x^2 **(B)** the coefficient of x

   (C) the y-intercept of the graph of $y = f(x)$

   (D) one of the x-intercepts of the graph of $y = f(x)$

   (E) the mean of the x-intercepts of the graph of $y = f(x)$

22. For each positive integer n, let $S(n)$ denote the sum of the digits of n. For how many values of n is $n + S(n) + S(S(n)) = 2007$?

   (A) 1   **(B)** 2   **(C)** 3   **(D)** 4   **(E)** 5

23. Square $ABCD$ has area 36, and $\overline{AB}$ is parallel to the x-axis. Vertices A, B, and C are on the graphs of $y = \log_a x$, $y = 2\log_a x$, and $y = 3\log_a x$, respectively. What is a?

   (A) $\sqrt[6]{3}$ **(B)** $\sqrt{3}$ **(C)** $\sqrt[3]{6}$ **(D)** $\sqrt{6}$ **(E)** 6

24. For each integer $n > 1$, let $F(n)$ be the number of solutions of the equation $\sin x = \sin nx$ on the interval $[0, \pi]$. What is $\sum_{n=2}^{2007} F(n)$?

    (A) 2,014,524   **(B)** 2,015,028   **(C)** 2,015,033   **(D)** 2,016,532

    (E) 2,017,033

25. Call a set of integers *spacy* if it contains no more than one out of any three consecutive integers. How many subsets of $\{1, 2, 3, \ldots, 12\}$, including the empty set, are spacy?

    (A) 121   **(B)** 123   **(C)** 125   **(D)** 127   **(E)** 129

2007 AIME

The AIME will be held on Tuesday, March 13, 2007, with the alternate on Wednesday, March 28, 2007. It is a 15-question, 3-hour, integer-answer exam. You will be invited to participate only if you score 120 or above, or finish in the top 1% of the AMC 10, or if you score 100 or above or finish in the top 5% of the AMC 12. Top-scoring students on the AMC 10/12/AIME will be selected to take the USA Mathematical Olympiad (USAMO) on April 24 and 25, 2007. The best way to prepare for the AIME and USAMO is to study previous exams. Copies may be ordered as indicated below.

PUBLICATIONS

A complete listing of current publications, with ordering instructions, is at our web site:
www.unl.edu/amc.

2007

AMC 12 – CONTEST A
DO NOT OPEN UNTIL
TUESDAY, February 6, 2007

**Administration On An Earlier Date Will Disqualify
Your School's Results**

1. All information (Rules and Instructions) needed to administer this exam is contained in the TEACHERS' MANUAL, which is outside of this package. **PLEASE READ THE MANUAL BEFORE February 6.** Nothing is needed from inside this package until February 6.

2. Your PRINCIPAL or VICE PRINCIPAL must sign the Certification Form found in the Teachers' Manual.

3. The Answer Forms must be mailed by First Class mail to the AMC no later than 24 hours following the examination.

4. *The publication, reproduction or communication of the problems or solutions of this test during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination during this period via copier, telephone, email, World Wide Web or media of any type is a violation of the competition rules.*