AMERICAN MATHEMATICS COMPETITIONS

16th ANNUAL
AMERICAN INVITATIONAL MATHEMATICS EXAMINATION
(AIME)
TUESDAY, MARCH 17, 1998

Sponsored by

Mathematical Association of America
Society of Actuaries   Mu Alpha Theta
National Council of Teachers of Mathematics
Casualty Actuarial Society   American Statistical Association
American Mathematical Association of Two-Year Colleges
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1. DO NOT OPEN THIS BOOKLET UNTIL TOLD TO DO SO BY YOUR PROCTOR.

2. This is a 15-question, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers; i.e., there is neither partial credit nor a penalty for wrong answers.

3. No aids other than scratch paper, graph paper, ruler, compass, and protractor are permitted. In particular, **calculators are <u>not</u> permitted.**

4. A combination of the AIME and AHSME scores is used to determine eligibility for participation in the U.S.A. Mathematical Olympiad (USAMO). The USAMO will be given on TUESDAY, April 28, 1998.

5. Record all your answers, and certain other information, on the AIME answer form. Only the answer form will be collected from you.

1. For how many values of k is 12^{12} the least common multiple of the positive integers 6^6, 8^8, and k?

2. Find the number of ordered pairs (x, y) of positive integers that satisfy $x \leq 2y \leq 60$ and $y \leq 2x \leq 60$.

3. The graph of $y^2 + 2xy + 40|x| = 400$ partitions the plane into several regions. What is the area of the bounded region?

4. Nine tiles are numbered 1, 2, 3, ..., 9, respectively. Each of three players randomly selects and keeps three of the tiles, and sums those three values. The probability that all three players obtain an odd sum is m/n, where m and n are relatively prime positive integers. Find $m + n$.

5. Given that $A_k = \dfrac{k(k-1)}{2} \cos \dfrac{k(k-1)\pi}{2}$, find $|A_{19} + A_{20} + \cdots + A_{98}|$.

6. Let $ABCD$ be a parallelogram. Extend $\overline{DA}$ though A to a point P, and let $\overline{PC}$ meet $\overline{AB}$ at Q and $\overline{DB}$ at R. Given that $PQ = 735$ and $QR = 112$, find RC.

7. Let n be the number of ordered quadruples (x_1, x_2, x_3, x_4) of positive odd integers that satisfy $\displaystyle\sum_{i=1}^{4} x_i = 98$. Find $\dfrac{n}{100}$.

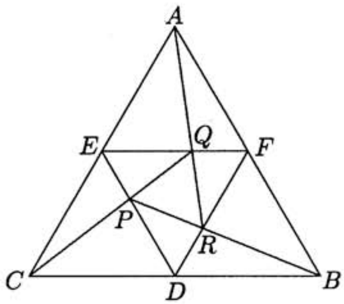
8. Except for the first two terms, each term of the sequence $1000,\ x,\ 1000 - x,\ \ldots$ is obtained by subtracting the preceding term from the one before that. The last term of the sequence is the first negative term encountered. What positive integer x produces a sequence of maximum length?

9. Two mathematicians take a morning coffee break each day. They arrive at the cafeteria independently, at random times between 9 a.m. and 10 a.m., and stay for exactly m minutes. The probability that either one arrives while the other is in the cafeteria is 40%, and $m = a - b\sqrt{c}$, where a, b, and c are positive integers, and c is not divisible by the square of any prime. Find $a + b + c$.

10. Eight spheres of radius 100 are placed on a flat surface so that each sphere is tangent to two others and their centers are the vertices of a regular octagon. A ninth sphere is placed on the flat surface so that it is tangent to each of the other eight spheres. The radius of this last sphere is $a + b\sqrt{c}$, where a, b, and c are positive integers, and c is not divisible by the square of any prime. Find $a + b + c$.

11. Three of the edges of a cube are $\overline{AB}$, $\overline{BC}$, and $\overline{CD}$, and $\overline{AD}$ is an interior diagonal. Points P, Q, and R are on $\overline{AB}$, $\overline{BC}$, and $\overline{CD}$, respectively, so that $AP = 5$, $PB = 15$, $BQ = 15$, and $CR = 10$. What is the area of the polygon that is the intersection of plane PQR and the cube?

12. Let ABC be equilateral, and D, E, and F be the midpoints of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. There exist points P, Q, and R on $\overline{DE}$, $\overline{EF}$, and $\overline{FD}$, respectively, with the property that P is on $\overline{CQ}$, Q is on $\overline{AR}$, and R is on $\overline{BP}$. The ratio of the area of triangle ABC to the area of triangle PQR is $a + b\sqrt{c}$, where a, b, and c are integers, and c is not divisible by the square of any prime. What is $a^2 + b^2 + c^2$?



13. If $\{a_1, a_2, a_3, \ldots, a_n\}$ is a set of real numbers, indexed so that $a_1 < a_2 < a_3 < \ldots < a_n$, its *complex power sum* is defined to be $a_1 i + a_2 i^2 + a_3 i^3 + \cdots + a_n i^n$, where $i^2 = -1$. Let S_n be the sum of the complex power sums of all nonempty subsets of $\{1, 2, \ldots, n\}$. Given that $S_8 = -176 - 64i$ and $S_9 = p + qi$, where p and q are integers, find $|p| + |q|$.

14. An $m \times n \times p$ rectangular box has half the volume of an $(m+2) \times (n+2) \times (p+2)$ rectangular box, where m, n, and p are integers, and $m \leq n \leq p$. What is the largest possible value of p?

15. Define a *domino* to be an ordered pair of *distinct* positive integers. A *proper sequence* of dominos is a list of distinct dominos in which the first coordinate of each pair after the first equals the second coordinate of the immediately preceding pair, and in which (i, j) and (j, i) do not *both* appear for any i and j. Let D_{40} be the set of all dominos whose coordinates are no larger than 40. Find the length of the longest proper sequence of dominos that can be formed using the dominos of D_{40}.

SOLUTIONS

A 1998 Solutions Pamphlet will be sent to exam managers within a few weeks.

WRITE TO US!

Correspondence about the problems and solutions for this AIME should be addressed to:

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1998 USAMO

The USA Mathematical Olympiad is a 6-question, 6-hour, essay-type examination. The USAMO will be held on **TUESDAY, APRIL 28, 1998**. Your teacher has more details on who qualifies for the USAMO in the AHSME or AIME Teachers' Manuals. The best way to prepare for the USAMO is to study the exams from previous years and to review the contents of the ARBELOS. Copies may be ordered as indicated below.

PUBLICATIONS

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FOREIGN ORDERS: Do **NOT** prepay; an invoice will be sent.

Examinations: Each price is for **one** copy of an exam **and** its solutions for one year. Specify the years you want and how many copies of each. All prices effective to September 1, 1998.

- **AHSME** 1983-98, $1 per copy per year.
- **AIME** 1983-98, $2 per copy per year.
- USA and International Mathematical **Olympiads** (together), 1976-97, $5 per copy per year.
- National **Summary** of Results and Awards, 1980-98, $10 per copy per year.
- Problem Book I, AHSMEs 1950-60, $9.00
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The Arbelos

- Short articles and challenging problems recommended especially for AIME and USAMO qualifiers. Six volumes, $8.00 each.