AMERICAN MATHEMATICS COMPETITIONS

11th ANNUAL
AMERICAN INVITATIONAL MATHEMATICS EXAMINATION
(AIME)
THURSDAY, APRIL 1, 1993

Sponsored by

Mathematical Association of America
Society of Actuaries   Mu Alpha Theta
National Council of Teachers of Mathematics
Casualty Actuarial Society   American Statistical Association
American Mathematical Association of Two-Year Colleges
American Mathematical Society

1. DO NOT OPEN THIS BOOKLET UNTIL TOLD TO DO SO BY YOUR PROCTOR.
2. This is a 15-question, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers; i.e., there is neither partial credit nor a penalty for wrong answers.
3. Only scratch paper, graph paper, ruler, compass, protractor, #2 pencil and eraser are permitted.
   Calculators, slide rules and math tables are not permitted.
4. Please print the following:

Name: ______________________________ Last ______________ First ______________ Middle initial

Home address: ______________________________ Street Address

______________ City ______________ State or Province ______________ Zip or Postcode

______________ Home Phone including Area Code ______ Sex (M or F) ______ Your age

______________ Full Name of School ______ Grade Level (e.g.,11)

5. Citizenship Status: USA Citizen_____ Permanent Resident_____ Other_____
   If other, explain: ______________________________
6. My score on the 1993 AHSME was ☐   I took the 1993 AHSME on (date): ______________
7. A combination of the AIME and AHSME scores is used to determine eligibility for participation in the U. S. A. Mathematical Olympiad (USAMO). The USAMO will be given on THURSDAY, April 29, 1993. Please check one box:
   If I qualify for the USAMO, I agree to take it. YES ☐ NO ☐
   (*Your school must also agree to administer the USAMO before you can take it.*)
8. Record all your answers, and certain other information, on the AIME answer form. Your Examination Manager will instruct you how to complete the form. Only the answer form and this cover will be collected from you.

1. How many even integers between 4000 and 7000 have four different digits?

2. During a recent campaign for office, a candidate made a tour of a country which we assume lies in a plane. On the first day of the tour he went east, on the second day he went north, on the third day west, on the fourth day south, on the fifth day east, etc. If the candidate went $n^2/2$ miles on the n^{th} day of his tour, how many miles was he from his starting point at the end of the 40^{th} day?

3. The table below displays some of the results of last summer's Frostbite Falls Fishing Festival, showing how many contestants caught n fish for various values of n.

n	0	1	2	3		13	14	15
number of contestants who caught n fish	9	5	7	23		5	2	1

In the newspaper story covering the event, it was reported that
a) the winner caught 15 fish;
b) those who caught 3 or more fish averaged 6 fish each;
c) those who caught 12 or fewer fish averaged 5 fish each.
What was the total number of fish caught during the festival?

4. How many ordered four-tuples of integers (a, b, c, d) with $0 < a < b < c < d < 500$ satisfy $a + d = b + c$ and $bc - ad = 93$?

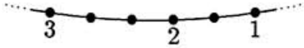
5. Let $P_0(x) = x^3 + 313x^2 - 77x - 8$. For integers $n \geq 1$, define $P_n(x) = P_{n-1}(x - n)$. What is the coefficient of x in $P_{20}(x)$?

6. What is the smallest positive integer that can be expressed as the sum of nine consecutive integers, the sum of ten consecutive integers, and the sum of eleven consecutive integers?

7. Three numbers, a_1, a_2, a_3, are drawn randomly and without replacement from the set $\{1, 2, 3, \ldots, 1000\}$. Three other numbers, b_1, b_2, b_3, are then drawn randomly and without replacement from the remaining set of 997 numbers. Let p be the probability that, after a suitable rotation, a brick of dimensions $a_1 \times a_2 \times a_3$ can be enclosed in a box of dimensions $b_1 \times b_2 \times b_3$, with the sides of the brick parallel to the sides of the box. If p is written as a fraction in lowest terms, what is the sum of the numerator and denominator?

8. Let S be a set with six elements. In how many different ways can one select two not necessarily distinct subsets of S so that the union of the two subsets is S? The order of selection does not matter; for example, the pair of subsets $\{a, c\}$, $\{b, c, d, e, f\}$ represents the same selection as the pair $\{b, c, d, e, f\}$, $\{a, c\}$.

9. Two thousand points are given on a circle. Label one of the points 1. From this point, count 2 points in the clockwise direction and label this point 2. From the point labeled 2, count 3 points in the clockwise direction and label this point 3. (See figure.) Continue this process until the labels $1, 2, 3, \ldots, 1993$ are all used. Some of the points on the circle will have more than one label and some points will not have a label. What is the smallest integer that labels the same point as 1993?



10. Euler's formula states that for a convex polyhedron with V vertices, E edges, and F faces, $V-E+F=2$. A particular convex polyhedron has 32 faces, each of which is either a triangle or a pentagon. At each of its V vertices, T triangular faces and P pentagonal faces meet. What is the value of $100P+10T+V$?

11. Alfred and Bonnie play a game in which they take turns tossing a fair coin. The winner of a game is the first person to obtain a head. Alfred and Bonnie play this game several times with the stipulation that the loser of a game goes first in the next game. Suppose that Alfred goes first in the first game, and that the probability that he wins the sixth game is m/n, where m and n are relatively prime positive integers. What are the last three digits of $m+n$?

12. The vertices of $\triangle ABC$ are $A=(0,0)$, $B=(0,420)$, and $C=(560,0)$. The six faces of a die are labeled with two A's, two B's, and two C's. Point $P_1=(k,m)$ is chosen in the interior of $\triangle ABC$, and points P_2, P_3, $P_4, \ldots$ are generated by rolling the die repeatedly and applying the rule: If the die shows label L, where $L \in \{A, B, C\}$, and P_n is the most recently obtained point, then P_{n+1} is the midpoint of $\overline{P_nL}$. Given that $P_7=(14,92)$, what is $k+m$?

13. Jenny and Kenny are walking in the same direction, Kenny at 3 feet per second and Jenny at 1 foot per second, on parallel paths that are 200 feet apart. A tall circular building 100 feet in diameter is centered midway between the paths. At the instant when the building first blocks the line of sight between Jenny and Kenny, they are 200 feet apart. Let t be the amount of time, in seconds, before Jenny and Kenny can see each other again. If t is written as a fraction in lowest terms, what is the sum of the numerator and denominator?

14. A rectangle that is inscribed in a larger rectangle (with one vertex on each side) is called *unstuck* if it is possible to rotate (however slightly) the smaller rectangle about its center within the confines of the larger. Of all the rectangles that can be inscribed unstuck in a 6 by 8 rectangle, the smallest perimeter has the form $\sqrt{N}$, for a positive integer N. Find N.

15. Let $\overline{CH}$ be an altitude of $\triangle ABC$. Let R and S be the points where the circles inscribed in triangles ACH and BCH are tangent to $\overline{CH}$. If $AB = 1995$, $AC = 1994$, and $BC = 1993$, then RS can be expressed as m/n, where m and n are relatively prime positive integers. Find $m + n$.

SOLUTIONS

A 1993 Solutions Pamphlet will be sent to exam managers within a few weeks.

WRITE TO US!

Questions and comments about the problems and solutions for this AIME (but **not** requests for the Solutions Pamphlet) should be addressed to:

Prof Elgin H Johnston, AIME Chairman
Department of Mathematics
Iowa State University, Ames, IA 50011 USA

Comments about administrative arrangements and orders for any publications listed below should be addressed to:

Prof Walter E Mientka, AMC Executive Director
Department of Mathematics and Statistics
University of Nebraska, Lincoln, NE 68588-0658 USA

1993 USAMO

The USA Mathematical Olympiad is a 5-question, $3\frac{1}{2}$ hour, essay-type examination. The USAMO will be held on THURSDAY, April 29, 1993. Your teacher has more details on who qualifies for the USAMO in the AHSME or AIME Teachers' Manuals. The best way to prepare for the USAMO is to study the exams from previous years and to review the contents of the ARBELOS. Copies may be ordered as indicated below.

PUBLICATIONS

MINIMUM ORDER: $5 (before handling fee), US FUNDS ONLY. Canada and US orders must be prepaid. Orders are mailed 4th class, unless you specify 1st class, in which case add $3.00 or 20% of total order, whichever is larger, with a maximum of $15.00. Make checks payable to the American Mathematics Competitions.

FOREIGN ORDERS: do **NOT** prepay; an invoice will be sent.

Examinations: Each price is for **one** copy of an exam **and** its solutions for one year. Specify the years you want and how many copies of each. All prices effective to September 1, 1993.

- **AJHSME** (Junior High Exam), 1985-1992, $1 per copy per year.
- **AHSME** 1980-93, $1 per copy per year.
- **AIME** 1983-93, $2 per copy per year.
- USA and International Mathematical **Olympiads** (together), 1976-92, $4 per copy per year.
- National **Summary** of Results and Awards, 1980-92, $4 per copy per year.

Books (Exams and solutions):

- Contest Problem Book I, AHSMEs 1950-60, $8.50.
- Contest Problem Book II, AHSMEs 1961-65, $8.50.
- Contest Problem Book III, AHSMEs 1966-72, $10.00.
- Contest Problem Book IV, AHSMEs 1973-82, $11.00.
- USA Mathematical Olympiads, 1972-86, $13.00.
- International Mathematical Olympiads, 1959-77, $10.00.
- International Mathematical Olympiads, 1978-85, $11.00.

Journal

- The ARBELOS (short articles and challenging problems); recommended especially for AIME and USAMO qualifiers. Five volumes plus a Geometry volume, $7.00 each.