AMERICAN MATHEMATICS COMPETITIONS

4th ANNUAL
AMERICAN INVITATIONAL MATHEMATICS EXAMINATION (AIME)

TUESDAY, MARCH 18, 1986

Sponsors:
MATHEMATICAL ASSOCIATION OF AMERICA
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INSTRUCTIONS

1. *Do not open* this book until told to do so.
2. This is a 15 question, 3 hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers; i.e., neither partial credit nor a penalty for wrong answers will be given.
3. All your answers, and certain other information, are to be recorded on a computer card. Your Examination Manager will instruct you how to fill out the card after you have finished with these instructions. Only the computer card and this cover sheet will be collected from you.
4. Scratch paper, graph paper, ruler, compass and eraser are permitted. *Calculators and slide rules are not permitted.*
5. Please print the following:

Last Name ____ First Name ____ Middle Initial ____

Home Address ____

City ____ State or Province ____ Zip or Postcode ____

Home Phone including Area Code ____ Sex (M or F) ____ Your age ____

Full Name of School ____ Grade Level (e.g., 11) ____

6. My score on the 1986 AHSME was ☐
   I took the 1986 AHSME on ____ (date)
7. This AIME is the qualifying examination for the U.S.A. Mathematical Olympiad (USAMO) to be given on April 22, 1986. Please check one box:
   If I qualify for the USAMO, I agree to take it. YES ☐ NO ☐

   Your school must also agree to administer the USAMO before you can take it.

1. What is the sum of the solutions of the equation $\sqrt[4]{x} = \dfrac{12}{7 - \sqrt[4]{x}}$?

2. Evaluate the product
$(\sqrt{5}+\sqrt{6}+\sqrt{7})\ (\sqrt{5}+\sqrt{6}-\sqrt{7})\ (\sqrt{5}-\sqrt{6}+\sqrt{7})\ (-\sqrt{5}+\sqrt{6}+\sqrt{7})$.

3. If $\tan x + \tan y = 25$ and $\cot x + \cot y = 30$, what is $\tan(x + y)$?

4. Determine $3x_4 + 2x_5$ if x_1, x_2, x_3, x_4 and x_5 satisfy the system of equations given below:

$$\begin{aligned} 2x_1 + x_2 + x_3 + x_4 + x_5 &= 6 \\ x_1 + 2x_2 + x_3 + x_4 + x_5 &= 12 \\ x_1 + x_2 + 2x_3 + x_4 + x_5 &= 24 \\ x_1 + x_2 + x_3 + 2x_4 + x_5 &= 48 \\ x_1 + x_2 + x_3 + x_4 + 2x_5 &= 96. \end{aligned}$$

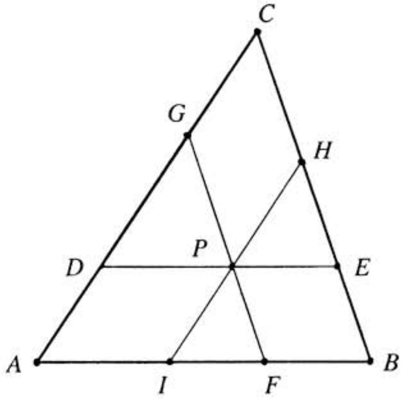
5. What is the largest positive integer n for which $n^3 + 100$ is divisible by $n + 10$?

6. The pages of a book are numbered 1 through n. When the page numbers of the book were added, one of the page numbers was mistakenly added twice, resulting in the incorrect sum of 1986. What was the number of the page that was added twice?

7. The increasing sequence 1, 3, 4, 9, 10, 12, 13, . . . consists of all those positive integers which are powers of 3 or sums of distinct powers of 3. Find the 100^{th} term of this sequence (where 1 is the 1^{st} term, 3 is the 2^{nd} term, and so on).

8. Let S be the sum of the base 10 logarithms of all of the proper divisors of 1,000,000. (By a proper divisor of a natural number we mean a positive integral divisor other than 1 and the number itself.) What is the integer nearest to S?

9. In $\triangle ABC$ shown below, $AB = 425$, $BC = 450$ and $CA = 510$. Moreover, P is an interior point chosen so that the segments DE, FG and HI are each of length d, contain P, and are parallel to the sides AB, BC and CA, respectively. Find d.



10. In a parlor game the "magician" asks one of the participants to think of a three-digit number (abc), where a, b and c represent digits in base 10 in the order indicated. Then the magician asks this person to form the numbers (acb), (bac), (bca), (cab) and (cba), to add these five numbers, and to reveal their sum, N. If told the value of N, the magician can identify the original number, (abc). Play the role of the magician and determine (abc) if $N = 3194$.

11. The polynomial $1 - x + x^2 - x^3 + \cdots + x^{16} - x^{17}$ may be written in the form $a_0 + a_1y + a_2y^2 + a_3y^3 + \cdots + a_{16}y^{16} + a_{17}y^{17}$, where $y = x + 1$ and the a_i's are constants. Find the value of a_2.

12. Let the sum of a set of numbers be the sum of its elements. Let S be a set of positive integers, none greater than 15. Suppose no two disjoint subsets of S have the same sum. What is the largest sum a set S with these properties can have?

13. In a sequence of coin tosses one can keep a record of the number of instances when a tail is immediately followed by a head, a head is immediately followed by a head, etc. We denote these by TH, HH, etc. For example, in the sequence $HHTTHHHHTHHTTTT$ of 15 coin tosses we observe that there are five HH, three HT, two TH and four TT subsequences. How many different sequences of 15 coin tosses will contain exactly two HH, three HT, four TH and five TT subsequences?

14. The shortest distances between an interior diagonal of a rectangular parallelpiped (box), P, and the edges it does not meet are $2\sqrt{5}$, $30/\sqrt{13}$ and $15/\sqrt{10}$. Determine the volume of P.

15. Let $\triangle ABC$ be a right triangle in the xy-plane with the right angle at C. Given that the length of the hypotenuse AB is 60, and that the medians through A and B lie along the lines $y = x + 3$ and $y = 2x + 4$, respectively, find the area of $\triangle ABC$.

Students and teachers with questions or comments about this AIME may write to:

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Questions about administrative arrangements for the AIME should be addressed to:

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Information about ordering past copies of the AIME and other examinations in the American Mathematics Competitions is found on the back cover of this examination.

This booklet should be kept by the participants since the statements of the problems is not repeated in the Solutions Pamphlet.

PUBLICATIONS LIST

The following publications are available for purchase by those interested in supplementary practice examination materials. Prices effective until October 1, 1986.

A. The American High School Mathematics Examination—Prior Examinations, 1972-1986, Spanish editions, 1978-86.
Specimen sets of prior examinations. Each set contains a question booklet and a solution pamphlet. 40¢ each set; specify years desired.

B. The American Invitational Mathematics Examination—1983-86 AIME and a solution pamphlet. 50¢ each set; specify years desired.

C. The U.S.A. and International Mathematical Olympiads, 1976-1985. These pamphlets contain the problems and solutions to the 1976-85 Olympiads. 50¢ each pamphlet; specify years desired.

D. *Contest Problem Book I* (at $6.50 each) contains AHSME questions and solutions for 1950-1960.
Contest Problem Book II (at $6.50 each) contains AHSME questions and solutions for 1961-1965.
Contest Problem Book III (at $7.50 each) contains AHSME questions and solutions for 1966-1972.
Contest Problem Book IV (at $8.50 each) contains AHSME questions and solutions for 1973-1982.
International Mathematical Olympiads (at $7.50 each) contains questions and solutions for 1959-1977.

E. The *Arbelos* (at $4.50 per subscription-5 issues per year) is a journal containing short articles and challenging problems for gifted students. Past issues 1982-83, 1983-84 and 1984-85 are also available at $4.50 each.

ORDER INFORMATION

TERMS: Minimum order of $4.00 is required.
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