AIME SOLUTIONS PAMPHLET

FOR STUDENTS AND TEACHERS

4th ANNUAL AMERICAN INVITATIONAL MATHEMATICS EXAMINATION 1986

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This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs. geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

It is hoped that teachers will find the time to share these solutions with their students, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution.

AMERICAN MATHEMATICS COMPETITIONS

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1. (Answer: 337)

   Let $y = \sqrt[4]{x}$. Then the equation may be written in the form

   $$y^2 - 7y + 12 = 0,$$

   whose roots are $y = 3$ and $y = 4$. Consequently, we obtain the x-values of 3^4 and 4^4, whose sum is 337.

2. (104)

   Repeated use of the identity $(x+y)(x-y) = x^2 - y^2$ leads to
   $(\sqrt{5}+\sqrt{6}+\sqrt{7})(\sqrt{5}+\sqrt{6}-\sqrt{7}) = (\sqrt{5}+\sqrt{6})^2 - (\sqrt{7})^2 = (11+2\sqrt{30}) - 7 = 4+2\sqrt{30}$,
   $(\sqrt{5}-\sqrt{6}+\sqrt{7})(-\sqrt{5}+\sqrt{6}+\sqrt{7}) = (\sqrt{7})^2 - (\sqrt{5}-\sqrt{6})^2 = 7 - (11-2\sqrt{30}) = -4+2\sqrt{30}$
   and $(4+2\sqrt{30})(-4+2\sqrt{30}) = (2\sqrt{30})^2 - 4^2 = 120 - 16 = 104$.

3. (150)

   Multiplying both sides of $\cot x + \cot y = 30$ by $\tan x \tan y$, we find that

   $$\tan y + \tan x = 30 \tan x \tan y.$$

   Since the left side of this equation is equal to 25, it follows that

   $$\tan x \tan y = \frac{25}{30} = \frac{5}{6}.$$

   Therefore,

   $$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{25}{1-\frac{5}{6}} = 150.$$

4. (181)

   Adding the five given equations, and then dividing both sides of the resulting equation by 6, yields

   (1) $$x_1 + x_2 + x_3 + x_4 + x_5 = 31.$$

   Subtracting (1) from the given fourth and fifth equations, we find that $x_4 = 17$ and $x_5 = 65$. Consequently, $3x_4 + 2x_5 = 51 + 130 = 181$.

5. (890)

By division we find that $n^3 + 100 = (n+10)(n^2 - 10n + 100) - 900$. Thus, if $n+10$ divides n^3+100, then it must also divide 900. Moreover, since n is maximized whenever $n+10$ is, and since the largest divisor of 900 is 900, we must have $n+10 = 900$. Therefore, $n = 890$.

6. (033)

Let k be the number of the page that was counted twice. Then, $0 < k < n+1$, and $1+2+\cdots+n+k$ is between $1+2+\cdots+n$ and $1+2+\cdots+n+(n+1)$. In other words, $n(n+1)/2 < 1986 < (n+1)(n+2)/2$; i.e.,

$$n(n+1) < 3972 < (n+1)(n+2).$$

By trial and error (clearly, n is a little larger than 60) we find that $n = 62$. Thus $k = 1986 - (62)(63)/2 = 1986 - 1953 = 33$.

7. (981)

If we use only the first six non-negative integral powers of 3, namely 1, 3, 9, 27, 81 and 243, then we can form only 63 terms, since

$$\binom{6}{1} + \binom{6}{2} + \cdots + \binom{6}{6} = 2^6 - 1 = 63.$$

Consequently, the next highest power of 3, namely 729, is also needed.

After the first 63 terms of the sequence the next largest ones will have 729 but not 243 as a summand. There are 32 of these, since $\binom{5}{0} + \binom{5}{1} + \cdots + \binom{5}{5} = 32$, bringing the total number of terms to 95. Since we need the 100^{th} term, we must next include 243 and omit 81. Doing so, we find that the 96^{th}, 97^{th}, ..., 100^{th} terms are: $729+243$, $729+243+1$, $729+243+3$, $729+243+3+1$ and $729+243+9 = 981$.

Alternate Solution. Note that a positive integer is a term of this sequence if and only if its base 3 representation consists only of 0's and 1's. Therefore, we can set up a one-to-one correspondence between the positive integers and the terms of this sequence by representing both with binary digits (0's and 1's), first in base 2 and then in base 3:

$$\begin{array}{rcrcrcr} 1 & = & 1_{(2)} & \longleftrightarrow & 1_{(3)} & = & 1 \\ 2 & = & 10_{(2)} & \longleftrightarrow & 10_{(3)} & = & 3 \\ 3 & = & 11_{(2)} & \longleftrightarrow & 11_{(3)} & = & 4 \\ 4 & = & 100_{(2)} & \longleftrightarrow & 100_{(3)} & = & 9 \\ 5 & = & 101_{(2)} & \longleftrightarrow & 101_{(3)} & = & 10 \\ & & & \vdots & & & \end{array}$$

This is a correspondence between the two sequences in the order given, that is, the k-th positive integer is made to correspond to the k-th sum (in increasing order) of distinct powers of 3. This is because, when the binary numbers are written in increasing order, they are still in increasing order when interpreted in any other base. (If you can explain why this is true when interpreted in base 10, you should be able to explain it in base 3 as well.)

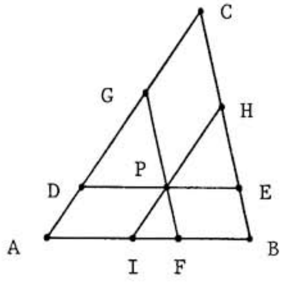
Therefore, to find the 100^{th} term of the sequence, we need only look at the 100^{th} line of the above correspondence:

$$100 = 1100100_{(2)} \longleftrightarrow 1100100_{(3)} = 981.$$

8. (141)

The number $1000000 = (2^6)(5^6)$ has $(6+1)(6+1) = 49$ distinct positive divisors. To see this, observe that they are all of the form $(2^i)(5^j)$; thus there are seven choices for i (0, 1, 2, ..., 6) and, independently, the same seven choices for j. Apart from 1000, the other 48 divisors form 24 pairs such that the product of each pair is 1000000. Since one of these pairs consists of the improper divisors 1 and 1000000, it follows that the product of all proper divisors of 1000000 is $(1000)(1000000)^{23}$ or 10^{141}. Moreover, since the sum, S, of the logarithms is equal to the logarithm of the product of these numbers, $S = 141$. The nearest integer to 141 is clearly 141.

9. (306)



As shown in the figure on the right,
$EH = BC - (BE + HC) = BC - (FP + PG) = 450 - d$.
In like manner, $GD = 510 - d$. Moreover, from the similarity of ΔDPG and ΔABC we have $DP/GD = AB/CA$. Hence

(1) $$DP = \frac{AB}{CA}\cdot GD = \frac{425}{510}(510 - d) = 425 - \frac{5}{6}d.$$

In like manner, since ΔPEH and ΔABC are similar, $PE/EH = AB/BC$. Hence

(2) $$PE = \frac{AB}{BC}\cdot EH = \frac{425}{450}(450 - d) = 425 - \frac{17}{18}d.$$

Since $d = DP + PE$, adding (1) and (2) we find that $d = 850 - \frac{16}{9}d$, from which $d = 306$.

10. (358)

Adding (abc) to N, and observing that each of the digits a, b and c appears exactly twice in each column, we are led to the equation

(1) $$N + (abc) = 222(a + b + c).$$

In view of this, the problem can be resolved by searching for an integral multiple of 222, say $222k$, which is larger than N (since $(abc) \neq 0$) and less than $N + 1000$ (since (abc) is a three-digit number), so that (1) is satisfied. That is, the sum of the digits of $222k - N$ must be k. If this holds, then $222k - N$ is the answer to the problem.

In our case, since $(14)(222) < 3194$ and $(19)(222) > 3194 + 1000$, the search is limited to $k = 15, 16, 17$ and 18. Of these only 16 works. Specifically, $(16)(222) - 3194 = 358$ and $3 + 5 + 8 = 16$.

Note. By also observing that $2N = 9(27a + 47b + 49c) + (a + b + c)$, one can slightly simplify the above computations. With the help of this observation one can also show that, whatever (abc) is, the magician can determine it uniquely.

11. (816)

Substituting $y-1$ for x, the given expression becomes

$$1 - (y-1) + (y-1)^2 - (y-1)^3 + \cdots + (y-1)^{16} - (y-1)^{17},$$

which may be written in the form

(1) $$1 + (1-y) + (1-y)^2 + (1-y)^3 + \cdots + (1-y)^{16} + (1-y)^{17}.$$

Note that each $(1-y)^k$ term in (1) will yield a y^2 term for $2 \le k \le 17$. More specifically, by the Binomial Theorem, each of the summands in (1) contributes $\binom{k}{2}$ or $k(k-1)/2$ to the coefficient of y^2. Therefore, the problem is equivalent to computing the sum

$$\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{17}{2}.$$

To this end, one may proceed directly (i.e., by calculating and adding the sixteen numbers 1, 3, 6, ..., 136); or use the result of the derivation

$$\sum_{k=2}^{n} \frac{k(k-1)}{2} = \sum_{k=1}^{n} \frac{k(k-1)}{2} = \frac{1}{2}\left(\sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} k \right)$$
$$= \frac{1}{2}\left(\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} \right)$$
$$= \frac{(n+1)\,n\,(n-1)}{6};$$

or use the more general formula

$$\sum_{k=m}^{n} \binom{k}{m} = \binom{n+1}{m+1}.$$

In any case, the desired sum is equal to 816.

12. (061)

First we show that S contains at most 5 elements. Suppose otherwise. Then S has at least $\binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4}$ or 56 subsets of 4 or fewer members. The sum of each of these subsets is at most 54 (since

$15+14+13+12 = 54$), hence, by the Pigeonhole Principle, at least two of these sums are equal. If the subsets are disjoint, we are done; if not, then the removal of the common element(s) yields the desired contradiction.

Next we attempt to construct such a 5-element set S, by choosing its elements as large as possible. Including 15, 14 and 13 in S leads to no contradiction, but if 12 is also in S, then (in view of $12+15 = 13+14$) the conditions on S would be violated. Hence we must omit 12. No contradiction results from letting 11 be a member of S, but then $10 \notin S$ since $10+15 = 11+14$, and $9 \notin S$ since $9+15 = 11+13$. So we must settle for 8 as the fifth element of S. Indeed, $S = \{8, 11, 13, 14, 15\}$ satisfies the conditions of the problem, yielding $8+11+13+14+15$ or 61 as the candidate for its solution.

Finally, to show that the maximum is indeed 61, suppose that the sum is more for another choice of S. Observe that this set must also contain 15, 14 and 13, for if even the smallest of them (13) is omitted, the maximum possible sum (62) is achievable only by including 10, 11 and 12, but then $15+11 = 14+12$. Having chosen 15, 14 and 13, we must exclude 12, as noted before. If 11 is included, then we are limited to the sum of 61 as above. If 11 is not included, then even by including 10 and 9 (which we can't) we could not surpass 61 since $15+14+13+10+9 = 61$. Consequently, 61 is indeed the maximum sum one can attain.

13. (560)

Think of such sequences of coin tosses as progressions of blocks of T's and H's, to be denoted by (T) and (H), respectively. Next note that each HT and TH subsequence signifies a transition from (H) to (T) and from (T) to (H), respectively. Since there should be three of the first kind and four of the second kind in each of the sequences of 15 coin tosses, one may conclude that each such sequence is of the form

(1) $$(T)(H)(T)(H)(T)(H)(T)(H).$$

Our next concern is the placement of T's and H's in their respective blocks, so as to assure that each sequence will have two HH and five TT subsequences. To this end, we will assume that each block in (1) initially contains only one member. Then, to satisfy the conditions of the problem, it will suffice to place 2 more H's into the (H)'s and 5 more T's into the (T)'s. Thus, to solve the problem, we must count the number of ways this can be accomplished.

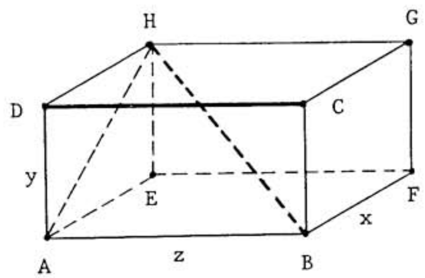
Recall that the number of ways to put p indistinguishable balls (the extra H's and T's in our case) into q distinguishable boxes (the (H)'s and (T)'s, distinguished by their order in the sequence) is given by the formula $\binom{p+q-1}{p}$. (Students who are not familiar with this fact should verify it.) In our case, it implies that the 2 H's can be placed in the 4 (H)'s in $\binom{2+4-1}{2}$ or 10 ways, and the 5 T's can be placed in the 4 (T)'s in $\binom{5+4-1}{5}$ or 56 ways. The desired answer is the product, 560, of these numbers.

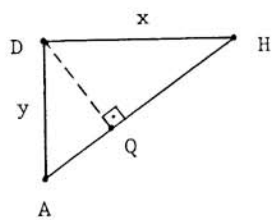
14. (750)

In order to find the volume of P, we will determine its dimensions, x, y and z. To this end, consider the rectangular parallelepiped shown in the first figure below, with vertices and side lengths as indicated. For definiteness, we labeled the box so that

(1) $\quad d(BH, CD) = 2\sqrt{5}, \quad d(BH, AE) = \frac{30}{\sqrt{13}} \quad \text{and} \quad d(BH, AD) = \frac{15}{\sqrt{10}},$

where, in general, d(RS, TU) denotes the distance between lines RS and TU.





Now the distance from BH to CD is equal to the distance from CD to the plane ABH, which is the same as the length of the perpendicular from D to the diagonal AH of rectangle AEHD. To see this, note that this perpendicular is also perpendicular to the plane ABH and the line CD. If one "slides" it over so that its top moves from D towards C, it eventually intersects line BH. This gives an equally long segment, which is perpendicular to both CD and BH, so its length is indeed the distance d(BH, CD). This distance is found via similar triangles, as shown in the second figure above, in which $DQ/DA = DH/AH$, and hence $DQ = xy/\sqrt{x^2+y^2}$. Treating the other distances similarly, in view of (1), this leads to the equations

$$(2) \qquad \frac{xy}{\sqrt{x^2+y^2}} = 2\sqrt{5}, \qquad \frac{yz}{\sqrt{y^2+z^2}} = \frac{30}{\sqrt{13}} \quad \text{and} \quad \frac{zx}{\sqrt{z^2+x^2}} = \frac{15}{\sqrt{10}}.$$

Upon squaring each equation in (2), taking reciprocals and simplifying, one arrives at the system

$$(3) \qquad \frac{1}{x^2}+\frac{1}{y^2} = \frac{1}{20}, \qquad \frac{1}{y^2}+\frac{1}{z^2} = \frac{13}{900} \quad \text{and} \quad \frac{1}{z^2}+\frac{1}{x^2} = \frac{2}{45}.$$

We solve (3) by first adding the equations therein, and then subtracting from $\frac{1}{2}$ times the resulting equation each of the original equations in (3). Thus we find that $1/x^2 = 1/25$, $1/y^2 = 1/100$ and $1/z^2 = 1/225$. From these, $x=5$, $y=10$ and $z=15$. Consequently, the volume of P is $xyz = 750$.

15. (400)

More generally, we will show that if $\triangle ABC$ is a right triangle with right angle at C, if $AB = 2r$ and if the acute angle between the medians emanating from A and B is θ, then

$$(1) \qquad \text{area}(\triangle ABC) = \frac{4}{3} r^2 \tan\theta.$$

In our case, $r = 60/2 = 30$ and $\tan\theta = 1/3$ (determined either by the standard formula for the tangent of the angle between two given lines, or by glancing at the first accompanying figure), so the answer to the problem is $(4/3)(30)^2(1/3)$ or 400.

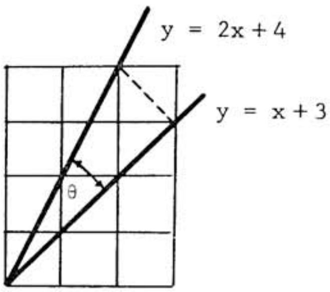
To establish (1), first note that

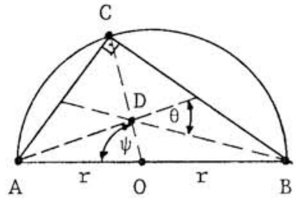
(2) $$\text{area}(\Delta ABC) = r^2 \sin\psi,$$

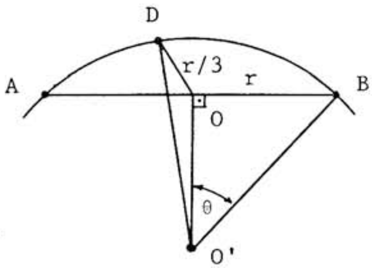
where $\psi = \measuredangle AOC$, as shown in the second figure below, where O is the mid-point of AB, D is the centroid of ΔABC and $CO = r$ (since ΔABC is a right triangle). Consequently, to prove (1), it suffices to verify that $\sin\psi = (4/3)\tan\theta$, which is equivalent to establishing that

(3) $$\cos\phi = -(4/3)\tan\theta,$$

where $\phi = \psi + 90^\circ$. This consideration leads us to the third figure below, where O' is the center of the circle through . . D and B, and $\measuredangle DOO' = \phi$.



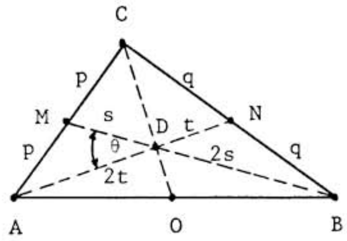




To prove (3), we will apply the Law of Cosines to $\Delta DOO'$. (The fact that $\measuredangle OO'B = \theta$ comes from the observation that $\measuredangle ADB = 180° - \theta$, and hence arc ADB has central angle 2θ ; $DO = r/3$ is a well-known fact concerning the centroid.) Noting that $DO' = BO' = r\csc\theta$ and $OO' = r\cot\theta$ (from $\Delta BOO'$), indeed,

$$\cos\phi = \frac{(r/3)^2 + (r\cot\theta)^2 - (r\csc\theta)^2}{2(r/3)(r\cot\theta)} = -\frac{4}{3}\tan\theta,$$

as was to be shown.



Alternate Solution. As in the diagram shown on the right, let M, N and O be the midpoints of the sides of $\triangle ABC$, D be its centroid, p, q, s and t be the distances indicated, and $\theta = \measuredangle ADM$. As in the previous solution, also observe that $\tan\theta = 1/3$, and hence $\sin\theta = 1/\sqrt{10}$. Then from $\text{area}(\triangle ABC) = 6\cdot\text{area}(\triangle ADM)$,

(1) $$\text{area}(\triangle ABC) = 6st\sin\theta = \frac{6}{\sqrt{10}}st,$$

and, since $\triangle ABC$ is a right triangle,

(2) $$\text{area}(\triangle ABC) = 2pq.$$

Moreover, by the Pythagorean Theorem, we find that

$$4p^2 + 4q^2 = AB^2 = 3600,$$
$$p^2 + 4q^2 = 9s^2,$$
$$4p^2 + q^2 = 9t^2.$$

With the help of these expressions from (1) and (2) it follows that

$$\begin{aligned}(\text{area}(\triangle ABC))^2 &= \frac{18}{5}s^2t^2\\ &= \frac{18}{5}\cdot\frac{p^2+4q^2}{9}\cdot\frac{4p^2+q^2}{9}\\ &= \frac{2}{45}((2p^2+2q^2)^2 + 9p^2q^2)\\ &= \frac{2}{45}((3600/2)^2 + \frac{9}{4}(\text{area}(\triangle ABC))^2)\\ &= 144{,}000 + \frac{1}{10}(\text{area}(\triangle ABC))^2.\end{aligned}$$

Consequently, $\frac{9}{10}(\text{area}(\triangle ABC))^2 = 144{,}000$, from which $\text{area}(\triangle ABC) = 400$.