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AMERICAN MATHEMATICS COMPETITIONS

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THE AKAMAI FOUNDATION

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AMERICAN INVITATIONAL MATHEMATICS EXAMINATION (AIME)

SOLUTIONS PAMPHLET

Tuesday, April 10, 2001

This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution. *Remember that reproduction of these solutions is prohibited by copyright.*

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1. (Answer: 816)
The possible pairs of consecutive digits are 01, 04, 09, 16, 25, 36, 49, 64, and 81. Choosing 16 as the leftmost pair of digits would yield 1649 as the greatest number with the requested property. Similarly, 25 would yield 25, 36 would yield 3649, 49 would yield 49, 64 would yield 649, and 81 would yield 81649. Of these, 81649 is the largest, and the leftmost three digits are 816.

2. (Answer: 298)
Let s be the number of students who study Spanish but not French, let f be the number of students who study French but not Spanish, and let b be the number of students who study both languages. It is given that $1600.8 < s+b < 1700.85$ and $600.3 < f+b < 800.4$; thus $1601 \leq s+b \leq 1700$ and $601 \leq f+b \leq 800$. Add the last pair of inequalities to obtain $2202 \leq s+f+2b \leq 2500$. Because $s+f+b = 2001$, it follows that $201 \leq b \leq 499$. The triples $(s,f,b) = (1400, 400, 201)$ and $(s,f,b) = (1201, 301, 499)$ show that $m = 201$ and $M = 499$, so $M - m = 298$.

OR

By the Inclusion-Exclusion Principle,

$$M = \lfloor 0.85(2001) \rfloor + \lfloor 0.40(2001) \rfloor - 2001 = 1700 + 800 - 2001 = 499$$

and

$$m = \lceil 0.80(2001) \rceil + \lceil 0.30(2001) \rceil - 2001 = 1601 + 601 - 2001 = 201,$$

so $M - m = 499 - 201 = 298$.

3. (Answer: 898)
For $n > 5$,

$$\begin{aligned} x_n &= x_{n-1} - x_{n-2} + x_{n-3} - x_{n-4} \\ &= (x_{n-2} - x_{n-3} + x_{n-4} - x_{n-5}) - x_{n-2} + x_{n-3} - x_{n-4} \\ &= -x_{n-5}. \end{aligned}$$

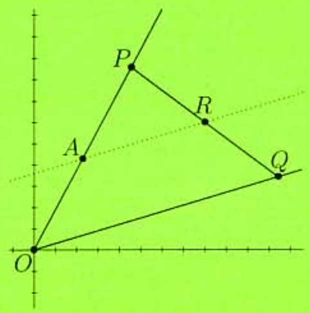
It follows that the sequence repeats in a cycle ten terms long. Hence

$$\begin{aligned} x_{531} + x_{753} + x_{975} &= x_1 + x_3 + x_5 \\ &= x_1 + x_3 + x_4 - x_3 + x_2 - x_1 \\ &= x_4 + x_2 \\ &= 523 + 375 = 898. \end{aligned}$$

OR

Using the theory of difference equations, a characteristic equation for the sequence is $x^4 = x^3 - x^2 + x - 1$ or $x^4 - x^3 + x^2 - x + 1 = 0$. Since $x^5 + 1 = (x+1)(x^4 - x^3 + x^2 - x + 1)$, we can conclude $x_n + x_{n-5} = 0$ and proceed as above.

4. (Answer: 067)
Let $O = (0, 0)$. The line through R that is parallel to $\overline{OQ}$ has equation $10y = 3x + 36$. This line meets $\overline{OP}$ at $A = \left(\frac{16}{7}, \frac{30}{7}\right)$. Because R is the midpoint of $\overline{PQ}$, it follows that A is the midpoint of $\overline{OP}$. Then $P = \left(\frac{32}{7}, \frac{60}{7}\right)$, and $PQ = 2PR = 2\sqrt{\left(\frac{24}{7}\right)^2 + \left(\frac{18}{7}\right)^2} = 2 \cdot \frac{6}{7} \cdot \sqrt{4^2 + 3^2} = \frac{60}{7}$. Thus $m + n = 67$.



OR

Let $P = (8t, 15t)$ and $Q = (10u, 3u)$. Because R is the midpoint of $\overline{PQ}$, it follows that

$$\begin{aligned} 8t + 10u &= 16 \text{ and} \\ 15t + 3u &= 12. \end{aligned}$$

The solution to this system is $t = \dfrac{4}{7}$ and $u = \dfrac{8}{7}$, so $P = \left(\dfrac{32}{7}, \dfrac{60}{7}\right)$, $Q = \left(\dfrac{80}{7}, \dfrac{24}{7}\right)$, and $PQ = \dfrac{1}{7}\sqrt{48^2 + 36^2} = \dfrac{12}{7}\sqrt{4^2 + 3^2} = \dfrac{60}{7}$. Thus $m + n = 67$.

5. (Answer: 253)

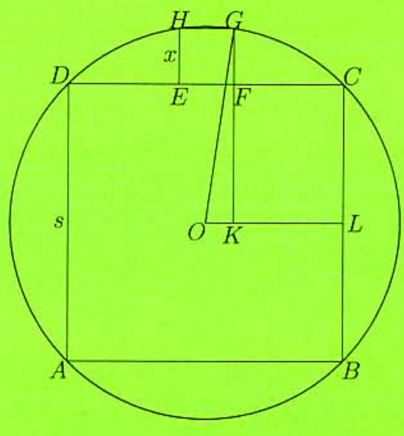
The set $\{4,5,9,14,23,37,60,97,157,254\}$ is a ten-element subset of $\{4,5,6,\ldots,254\}$ that does not have the triangle property. Let N be the smallest integer for which $\{4,5,6,\ldots,N\}$ has a ten-element subset that lacks the triangle property. Let $\{a_1,a_2,a_3,\ldots,a_{10}\}$ be such a subset, with $a_1 < a_2 < a_3 < \cdots < a_{10}$. Because none of its three-element subsets define triangles, the following must be true:

$$\begin{aligned} N &\geq a_{10} \\ &\geq a_9 + a_8 \\ &\geq (a_8 + a_7) + a_8 = 2a_8 + a_7 \\ &\geq 2(a_7 + a_6) + a_7 = 3a_7 + 2a_6 \\ &\geq 3(a_6 + a_5) + 2a_6 = 5a_6 + 3a_5 \\ &\geq 8a_5 + 5a_4 \\ &\geq 13a_4 + 8a_3 \\ &\geq 21a_3 + 13a_2 \\ &\geq 34a_2 + 21a_1 \\ &\geq 34 \cdot 5 + 21 \cdot 4 = 254 \end{aligned}$$

Thus the largest possible value of n is $N - 1 = 253$.

6. (Answer: 251)

Let O be the center of the circle, and represent the lengths of each side of the small square and the large square by x and s, respectively . Draw $\overline{OL}$ perpendicular to $\overline{BC}$ at L and $\overline{FK}$ perpendicular to $\overline{OL}$ at K. Then $GK = GF + FK = GF + CL = x + \frac{s}{2}$, $OK = x/2$, and the circle's radius is $(1/2)s\sqrt{2}$. Applying the Pythagorean Theorem to triangle OKG, we obtain $(x+\frac{s}{2})^2 + (\frac{x}{2})^2 = (\frac{s\sqrt{2}}{2})^2$. Expanding yields $x^2 + sx + \frac{s^2}{4} + \frac{x^2}{4} = \frac{s^2}{2}$ which leads to $5x^2 + 4sx - s^2 = 0$, or $(5x - s)(x + s) = 0$, so $x = s/5$. The ratio of the squares' areas is thus $1/25$, and $10n + m = 251$.



7. (Answer: 725)

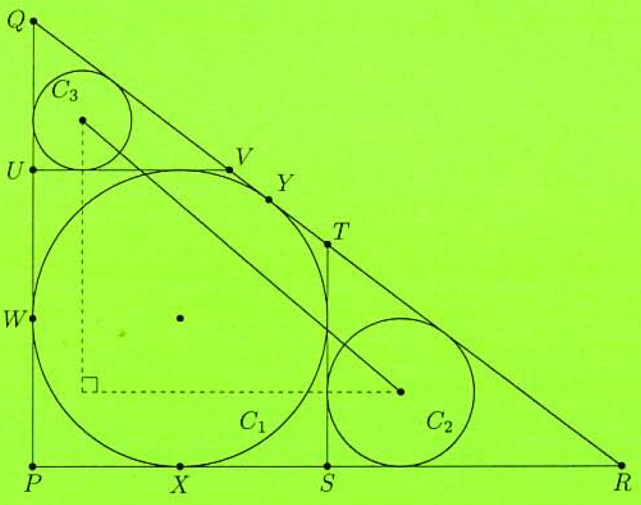
Let r_1, r_2, and r_3 be the radii of circles C_1, C_2, and C_3, respectively. The inradius of any triangle is twice the area divided by the perimeter, so $r_1 = 90 \cdot 120/(90+120+150) = 30$. Because $\triangle RST$ is similar to $\triangle RPQ$ and $RS = PR - 2r_1 = 60$, the similarity constant is $1/2$. Thus $r_2 = 15$. Similarly, $r_3 = 10$. If d is the distance between the centers of C_2 and C_3, then

$$\begin{aligned} d^2 &= (2r_1 + r_2 - r_3)^2 + (2r_1 + r_3 - r_2)^2 \\ &= 65^2 + 55^2 \\ &= 4225 + 3025 \\ &= 7250. \end{aligned}$$

Hence $n = 725$.

OR

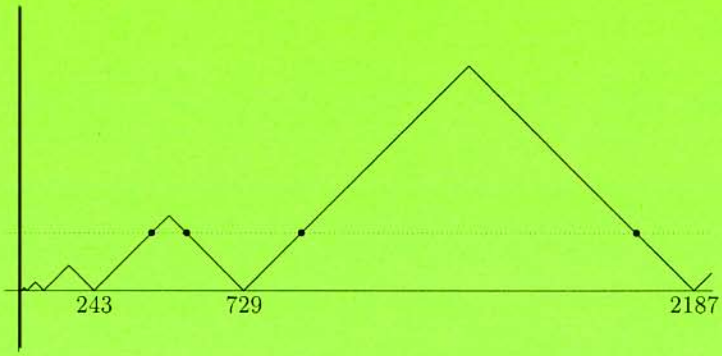
Let W, X, and Y be the points of tangency of circle C_1 to $\overline{PQ}$, $\overline{PR}$, and $\overline{RQ}$, respectively. Note that $PW = PX = r_1$. Then $RY = RX = 120 - r_1$, and $QY = QW = 90 - r_1$, from which we obtain $120 - r_1 + 90 - r_1 = 150$, and $r_1 = 30$. Assign coordinates so that $P = (0,0)$, $Q = (0,90)$, and $R = (120,0)$. Now $U = (0,60)$ and $S = (60,0)$. Because triangles UQV and STR are similar to triangle PQR with similarity constants $1/3$ and $1/2$, respectively, conclude that $r_3 = 10$ and $r_2 = 15$. Thus the centers of circles C_2 and C_3 have coordinates $(75,15)$ and $(10,70)$, respectively. Use the distance formula to find that $d^2 = 65^2 + 55^2 = 7250$.



8. (Answer: 429)
First calculate

$$f(2001) = 3f\left(\frac{2001}{3}\right) = 9f\left(\frac{2001}{9}\right) = \cdots = 729f\left(\frac{2001}{729}\right) = 729\left(1 - \frac{543}{729}\right) = 186.$$

For $1 \leq x \leq 3$, the graph of $y = f(x)$ consists of segments that join $(2,1)$ to $(1,0)$ and to $(3,0)$. The definition of f implies that $(3a, 3b)$ is on the graph of f whenever (a,b) is, so the positive x-axis and the graph of f form a sequence of isosceles right triangles, each a threefold magnification of its predecessor. Notice that 3^n is the altitude of the triangle whose left vertex is $(3^n, 0)$ and whose right vertex is $(3^{n+1}, 0)$. Because the line $y = 186$ intersects only those triangles whose altitudes are at least 186, the leftmost intersection point is found, as shown, in the triangle whose left vertex is $(243, 0)$ and whose right vertex is $(729, 0)$. The desired point is found on a segment of slope 1, so $x = 243 + 186 = 429$.



9. (Answer: 929)
Number the squares as shown. For i=1, 2, 4, and 5, let Q_i be the event that i is the upper left corner of a 2-by-2 red square, and let $p(E)$ be the probability that event E will occur. By the Inclusion-Exclusion Principle, the probability that the grid *does* have at least one 2-by-2 red square is

1	2	3
4	5	6
7	8	9

$$\begin{aligned}
&p(Q_1) + p(Q_2) + p(Q_4) + p(Q_5) \\
&-p(Q_1 \cap Q_2) - p(Q_1 \cap Q_4) - p(Q_2 \cap Q_5) - p(Q_4 \cap Q_5) - p(Q_1 \cap Q_5) - p(Q_2 \cap Q_4) \\
&+p(Q_1 \cap Q_2 \cap Q_5) + p(Q_1 \cap Q_2 \cap Q_4) + p(Q_1 \cap Q_4 \cap Q_5) + p(Q_2 \cap Q_4 \cap Q_5) \\
&-p(Q_1 \cap Q_2 \cap Q_4 \cap Q_5)
\end{aligned}$$

or

$$4\left(\frac{1}{2}\right)^4 - \left[4\left(\frac{1}{2}\right)^6 + 2\left(\frac{1}{2}\right)^7\right] + 4\left(\frac{1}{2}\right)^8 - \left(\frac{1}{2}\right)^9 = \frac{95}{512}.$$

The probability that the grid does *not* have at least one 2-by-2 red square is therefore $1 - 95/512 = 417/512$, so $m + n = 929$.

10. (Answer: 784)
Because

$$10^j - 10^i = 10^i(10^{j-i} - 1),$$

and $1001 = 7 \cdot 11 \cdot 13$ is relatively prime to 10^i, it is necessary to find i and j so that $10^{j-i} - 1$ is divisible by the primes 7, 11, and 13. Notice that 10^6 is the smallest power of 10 that leaves a remainder of 1 when divided by 7 or 13, and that 10^2 is the smallest power of 10 that leaves a remainder of 1 when divided by 11. Hence $10^i(10^{j-i} - 1)$ is divisible by 1001 if and only if $j - i = 6n$ for some positive integer n. Thus it is necessary to count the number of integer solutions to

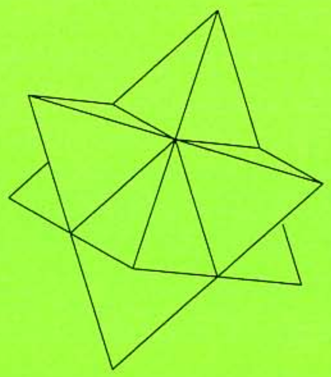
$$i + 6n = j, \text{ where } j \leq 99, i \geq 0, n > 0.$$

For each $n = 1, 2, 3, \ldots, 16$, there are $100 - 6n$ suitable values of i (and j), so the number of solutions is

$$94 + 88 + 82 + \cdots + 4 = 784.$$

11. (Answer: 341)
The probability P that the team has more wins than losses is the same as the probability that the team has more losses than wins, and hence $P = \frac{1}{2}(1-S)$, where S is the probability that Truncator has the same number of wins as losses. The probability of three wins and three losses is $\binom{6}{3}\left(\frac{1}{3}\right)^6$, the probability of two wins and two losses is $\binom{6}{2}\binom{4}{2}\left(\frac{1}{3}\right)^6$, the probability of one win and one loss is $\binom{6}{1}\binom{5}{1}\left(\frac{1}{3}\right)^6$, and the probability of no wins and no losses is $\left(\frac{1}{3}\right)^6$. Therefore $S = (20 + 90 + 30 + 1)\left(\frac{1}{3}\right)^6 = \frac{141}{729} = \frac{47}{243}$, and $P = \frac{1}{2} \cdot \left(1 - \frac{47}{243}\right) = \frac{98}{243}$. Thus $m + n = 98 + 243 = 341$.

12. (Answer: 101)
The diagram shows $\mathcal{P}_1$. Notice that $\mathcal{P}_0$ has 4 triangular faces, $\mathcal{P}_1$ has 24, and, inductively, $\mathcal{P}_i$ has $4 \cdot 6^i$. This expression therefore counts the small tetrahedra that are attached to $\mathcal{P}_i$ to form $\mathcal{P}_{i+1}$. The volume of each of these small tetrahedra is $(\frac{1}{8})^{i+1}$, and hence the volume of $\mathcal{P}_{i+1}$ is $4 \cdot 6^i(\frac{1}{8})^{i+1} = (\frac{1}{2})(\frac{3}{4})^i$ more than the volume of $\mathcal{P}_i$. In particular, the volume of $\mathcal{P}_3$ is



$$1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{3}{4}\right) + \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)^2 = \frac{69}{32}.$$

Thus $m + n = 101$.

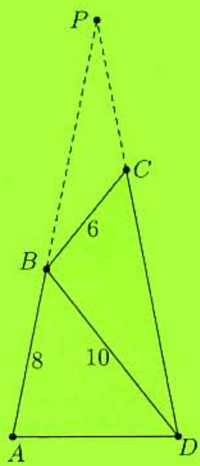
3. (Answer: 069)

Let P be the point where $\overline{AB}$ and $\overline{DC}$ intersect when extended. Since angles PCB and PBD are supplements of angles BCD and ABD, respectively, angles PCB and PBD are congruent, which implies that triangles PCB and PBD are similar. Now

$$\frac{PD-8}{PD} = \frac{PB}{PD} = \frac{CB}{BD} = \frac{6}{10},$$

so $PA = PD = 20$ and $PB = 12$. Then $\frac{PC}{12} = \frac{6}{10}$, so $PC = 7.2$ and $CD = 12.8 = 64/5$. Thus $m+n = 69$.

Query: Can you prove that B is between A and P and that C is between D and P?



4. (Answer: 840)

Because $|z| = 1, z = \cos\theta + i\sin\theta$, with $0 \le \theta < 360$. Now $\cos 28\theta - \cos 8\theta = 1$ and $\sin 28\theta - \sin 8\theta = 0$. From the latter equation, conclude that $28\theta + 8\theta = 180 + 360k$ for some integer k, so $\theta = 10k + 5$. It follows from the former equation that

$$-2\sin\frac{28\theta+8\theta}{2}\sin\frac{28\theta-8\theta}{2} = 1,$$

which is equivalent to $\sin 18\theta \sin 10\theta = -\frac{1}{2}$. Substitute $\theta = 10k+5$ to obtain $\sin(180k + 90)\sin(100k+50) = -\frac{1}{2}$, or $\sin(100k+50) = (-1)^{k+1}\cdot\frac{1}{2}$. When $k = 2m-1$ (that is, when k is odd) $\sin(200m - 50) = \frac{1}{2}$. Then $200m - 50 \equiv 30$ or $150 \pmod{360}$ yields $m \equiv 4$ or $1 \pmod 9$, $k \equiv 7$ or $1 \pmod{18}$, and $\theta \equiv 75$ or $15 \pmod{180}$. When $k = 2m$, similar reasoning leads to $\theta \equiv 165$ or $105 \pmod{180}$. Thus $\theta \equiv \pm 15 \pmod{90}$ and $\theta_2 + \theta_4 + \theta_6 + \theta_8 = \Sigma_{k=1}^{4}(90k - 15) = 840$.

OR

Let $\operatorname{cis}\theta$ denote $\cos\theta + i\sin\theta$. If z satisfies $z^{28} - z^8 - 1 = 0$, then $z^8(z^{20}-1) = 1$. Because $|z| = 1$, it follows that $|z^{20} - 1| = 1$ and $|z^{20}| = 1$. This can happen only if $z^{20} = \operatorname{cis}(\pm 60)$, in which case $z^{20} - 1 = \operatorname{cis}(\pm 120)$. Therefore $z^8 = \operatorname{cis}(\mp 120)$ and $z^4 = z^{20}/(z^8)^2 = \operatorname{cis}(\pm 300) = \operatorname{cis}(\mp 60)$, which implies that $z = \operatorname{cis}(90k \mp 15)$ for some integer k. Such z satisfy $z^{28} = \operatorname{cis}(\mp 60) = 1 + \operatorname{cis}(\mp 120) = 1 + z^8$. Thus the equation $z^{28} - z^8 - 1 = 0$ has eight solutions on the unit circle, namely $\theta_1 = 15$, $\theta_2 = 75$, $\theta_3 = 105$, $\theta_4 = 165$, $\theta_5 = 195$, $\theta_6 = 255$, $\theta_7 = 285$, and $\theta_8 = 345$. It follows that $\theta_2+\theta_4+\theta_6+\theta_8 = 840$.

OR

With $z = \cos\theta + i\sin\theta$, write the equation as $z^{18}(z^{10} - z^{-10}) = 1$, and notice that $z^{10} - z^{-10} = 2i\sin 10\theta$. Then, taking the absolute value of each side of the original equation, $2\sin 10\theta = \pm 1$, and thus z is a solution with $|z| = 1$ if and only if

$$z^{18} = \mp i \quad \text{and} \quad \sin 10\theta = \pm 1/2.$$

Hence the desired values of θ satisfy $18\theta \equiv 270$ or $90 \pmod{360}$ with $10\theta \equiv 30$ or $150 \pmod{360}$ in the first case and $10\theta \equiv 210$ or $330 \pmod{360}$ in the second. Thus

$$\theta \equiv 15 \pmod{20} \text{ and } \theta \equiv 3 \text{ or } 15 \pmod{36}$$

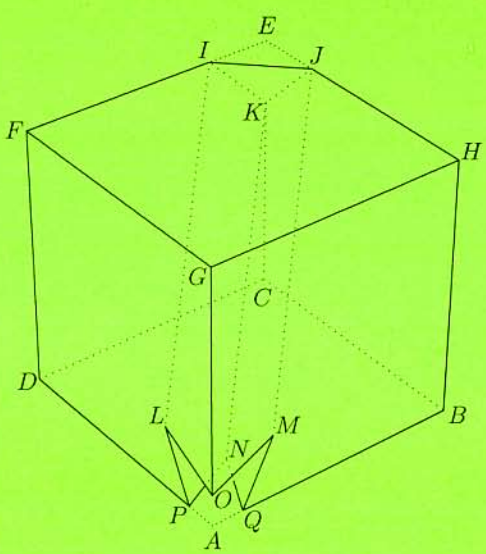
or

$$\theta \equiv 5 \pmod{20} \text{ and } \theta \equiv 21 \text{ or } 33 \pmod{36}.$$

The smallest positive solutions are 75 and 15 in the first case, and 165 and 105 in the second. Solutions are congruent modulo 180, so the solutions between 0 and 360 are 15, 75, 105, 165, 195, 255, 285, 345. Thus $\theta_2 + \theta_4 + \theta_6 + \theta_8 = 840$.

15. (Answer: 417)

Assign coordinates $A = (0,0,0)$, $B = (8,0,0)$, $C = (8,8,0)$, $D = (0,8,0)$, $I = (6,8,8)$, $J = (8,6,8)$, and $K = (8,8,6)$. The line through I that is parallel to $\overline{AE}$ can be described by $(x,y,z) = (6-t, 8-t, 8-t)$, so this line meets the cube again when $x = 0$, at $L = (0,2,2)$. By symmetry, the lines through J and K that are parallel to $\overline{AE}$ intersect the cube again at $M = (2,0,2)$ and $N = (2,2,0)$, respectively. It is straightforward to show that the plane determined by I, J, and L is described by the equation $2z = 2 + x + y$, so that the plane meets the z-axis at $O = (0,0,1)$. By symmetry, the tunnel intersects the x-axis at $Q = (1,0,0)$ and the y-axis at $P = (0,1,0)$. As the diagram shows, one end of the tunnel has a triangular opening IJK, while the other has a non-planar hexagonal opening $LOMQNP$. The surface of $\mathcal{S}$ consists of nine polygonal faces, three of each of three types. It is straightforward to show that the area of pentagon $IFGHJ$ and the area of hexagon $CDPNQB$ are both $8^2 - 2$. To find the area of pentagon $ILOMJ$, first obtain $IJ = 2\sqrt{2}$, $IL = JM = 6\sqrt{3}$, and $LO = OM = \sqrt{5}$. Then calculate the area of rectangle $ILMJ$ to be $12\sqrt{6}$ and the area of isosceles triangle LOM to be $\sqrt{6}$. Thus the total surface area of $\mathcal{S}$ is $3(62 + 62 + 13\sqrt{6}) = 372 + 39\sqrt{6}$, and $m + n + p = 417$.



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