AMERICAN MATHEMATICS COMPETITIONS

15th ANNUAL
AMERICAN INVITATIONAL MATHEMATICS EXAMINATION
(AIME)
THURSDAY, MARCH 20, 1997

Sponsored by

Mathematical Association of America
Society of Actuaries   Mu Alpha Theta
National Council of Teachers of Mathematics
Casualty Actuarial Society   American Statistical Association
American Mathematical Association of Two-Year Colleges
American Mathematical Society
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1. DO NOT OPEN THIS BOOKLET UNTIL TOLD TO DO SO BY YOUR PROCTOR.
2. This is a 15-question, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers; i.e., there is neither partial credit nor a penalty for wrong answers.
3. No aids other than scratch paper, graph paper, ruler, compass, and protractor are permitted. In particular, **calculators are <u>not</u> permitted.**
4. Please print the following:

Name: ______________________________
Last   First   Middle initial

Home address: ______________________________
Street Address

____
City   State or Province   Zip or Postcode

____
Home Phone including Area Code   Gender (M or F)   Your age

____
Full Name of School   Grade Level (e.g.,11)

5. Citizenship Status: USA Citizen____ *Permanent Resident____ Other____
   If other, explain: ______________________________
   *Permanent Resident means someone seeking citizenship and currently possessing a U.S.A. Immigration "green card".
6. A combination of the AIME and AHSME scores is used to determine eligibility for participation in the U.S.A. Mathematical Olympiad (USAMO). The USAMO will be given on THURSDAY, May 1, 1997.
7. Record all your answers, and certain other information, on the AIME answer form. Only the answer form and this cover will be collected from you.

1. How many of the integers between 1 and 1000, inclusive, can be expressed as the difference of the squares of two nonnegative integers?

2. The nine horizontal and nine vertical lines on an 8×8 checkerboard form r rectangles, of which s are squares. The number s/r can be written in the form m/n, where m and n are relatively prime positive integers. Find $m+n$.

3. Sarah intended to multiply a two-digit number and a three-digit number, but she left out the multiplication sign and simply placed the two-digit number to the left of the three-digit number, thereby forming a five-digit number. This number is exactly nine times the product Sarah should have obtained. What is the sum of the two-digit number and the three-digit number?

4. Circles of radii 5, 5, 8, and m/n are mutually externally tangent, where m and n are relatively prime positive integers. Find $m+n$.

5. The number r can be expressed as a four-place decimal $0.abcd$, where a, b, c, and d represent digits, any of which could be zero. It is desired to approximate r by a fraction whose numerator is 1 or 2 and whose denominator is an integer. The closest such fraction to r is $\frac{2}{7}$. What is the number of possible values for r?

6. Point B is in the exterior of the regular n-sided polygon $A_1A_2\ldots A_n$, and A_1A_2B is an equilateral triangle. What is the largest value of n for which A_n, A_1, and B are consecutive vertices of a regular polygon?

7. A car travels due east at $\frac{2}{3}$ mile per minute on a long, straight road. At the same time, a circular storm, whose radius is 51 miles, moves southeast at $\frac{1}{2}\sqrt{2}$ mile per minute. At time $t=0$, the center of the storm is 110 miles due north of the car. At time $t=t_1$ minutes, the car enters the storm circle, and at time $t=t_2$ minutes, the car leaves the storm circle. Find $\frac{1}{2}(t_1+t_2)$.

8. How many different 4×4 arrays whose entries are all 1's and -1's have the property that the sum of the entries in each row is 0 and the sum of the entries in each column is 0?

9. Given a nonnegative real number x, let $\langle x \rangle$ denote the fractional part of x; that is, $\langle x \rangle = x - \lfloor x \rfloor$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. Suppose that a is positive, $\left\langle a^{-1} \right\rangle = \left\langle a^2 \right\rangle$, and $2 < a^2 < 3$. Find the value of $a^{12} - 144a^{-1}$.

10. Every card in a deck has a picture of one shape — circle, square, or triangle, which is painted in one of three colors — red, blue, or green. Furthermore, each color is applied in one of three shades — light, medium, or dark. The deck has 27 cards, with every shape-color-shade combination represented. A set of three cards from the deck is called *complementary* if all of the following statements are true:
    i. Either each of the three cards has a different shape or all three of the cards have the same shape.
    ii. Either each of the three cards has a different color or all three of the cards have the same color.
    iii. Either each of the three cards has a different shade or all three of the cards have the same shade.

    How many different complementary three-card sets are there?

11. Let $x = \dfrac{\sum_{n=1}^{44} \cos n^\circ}{\sum_{n=1}^{44} \sin n^\circ}$. What is the greatest integer that does not exceed $100x$?

12. The function f defined by $f(x) = \dfrac{ax+b}{cx+d}$, where a, b, c, and d are nonzero real numbers, has the properties $f(19) = 19$, $f(97) = 97$, and $f(f(x)) = x$ for all values of x except $-d/c$. Find the unique number that is not in the range of f.

13. Let S be the set of points in the Cartesian plane that satisfy

$$\Big|\big||x|-2\big|-1\Big| + \Big|\big||y|-2\big|-1\Big| = 1.$$

    If a model of S were built from wire of negligible thickness, then the total length of wire required would be $a\sqrt{b}$, where a and b are positive integers and b is not divisible by the square of any prime number. Find $a+b$.

14. Let v and w be distinct, randomly chosen roots of the equation $z^{1997} - 1 = 0$. Let m/n be the probability that $\sqrt{2+\sqrt{3}} \le |v+w|$, where m and n are relatively prime positive integers. Find $m+n$.

15. The sides of rectangle $ABCD$ have lengths 10 and 11. An equilateral triangle is drawn so that no point of the triangle lies outside $ABCD$. The maximum possible area of such a triangle can be written in the form $p\sqrt{q} - r$, where p, q, and r are positive integers, and q is not divisible by the square of any prime number. Find $p+q+r$.

SOLUTIONS

A 1997 Solutions Pamphlet will be sent to exam managers within a few weeks.

WRITE TO US!

Correspondence about the problems and solutions for this AIME should be addressed to:

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Orders for any publications listed below should be addressed to:

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1997 USAMO

The USA Mathematical Olympiad (USAMO) will be held on Thursday, May 1, 1997. It is a 6-question, 6-hour, essay-type examination. Your teacher has more details on who qualifies for the USAMO in the AHSME or AIME teachers' Manuals. The best way to prepare for the AIME and USAMO is to study previous years of these exams. Copies may be ordered as indicated below.

PUBLICATIONS

MINIMUM ORDER: $5 (before postage and handling fee), U.S. FUNDS ONLY. **U.S.A. Only:** 4th Class (surface) Postage is free for prepaid orders. First Class/Air Mail Postage and Handling: add 20% of the total order, with a minimum of $3 and a maximum of $15. **Please note that if the correct First Class cost is not included, the order will be sent 4th class.** Schools with zip codes outside the continental U.S. should request First Class/Air Mail service. Delivery takes 4 to 12 weeks. **Canadian Schools:** Surface Postage and Handling add 15% of total order with a $3 min. and $20 max. Air Mail Postage and Handling add 20% with a $4 min. and $20 max. Make checks payable to the American Mathematics Competitions; or give Visa or MasterCard number and expiration date.

FOREIGN ORDERS: Do **NOT** prepay. A Proforma invoice will be sent including Postage and Handling Fees.

Examinations: Each price is for one copy of an exam and its solutions for one year. Specify the years you want and how many copies of each. All prices effective to September 1, 1997.

- **AHSME** 1983-97, $1 per copy per year.
- **AIME** 1983-97, $2 per copy per year.
- USA and International Mathematical **Olympiads** (together), 1983-96, $5 per copy per year.
- National **Summary** of Results and Awards, 1980-97, $10 per copy per year.
- Problem Book I, AHSMEs 1950-60, $8.00
- Problem Book II, AHSMEs 1961-65, $8.00
- Problem Book III, AHSMEs 1966-72, $13.50
- Problem Book IV, AHSMEs 1973-82, $13.50
- USA Mathematical Olympiad Book 1972-86, $16.00
- International Mathematical Olympiad Book I, 1959-77, $14.00
- International Mathematical Olympiad Book II, 1978-85, $11.50
- 1995 Olympiad problems (no solutions) from 23 countries, $10.00

The Arbelos

- Short articles and challenging problems recommended especially for AIME and USAMO qualifiers. Six volumes, $8.00 each.