AMERICAN MATHEMATICS COMPETITIONS

AIME SOLUTIONS PAMPHLET
FOR STUDENTS AND TEACHERS

9th ANNUAL
AMERICAN INVITATIONAL MATHEMATICS EXAMINATION
(AIME)

TUESDAY, MARCH 19, 1991

Sponsored by

Mathematical Association of America
Society of Actuaries   Mu Alpha Theta
National Council of Teachers of Mathematics
Casualty Actuarial Society   American Statistical Association
American Mathematical Association of Two-Year Colleges
American Mathematical Society

This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs. geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers will share these solutions with their students, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution. ***Remember that reproduction of these solutions is prohibited by copyright.***

Correspondence about the problems and solutions (but not requests for the Solutions Pamphlet) should be addressed to:

Professor Elgin H Johnston, AIME Chairman
Department of Mathematics
Iowa State University
Ames, IA 50011 USA

Order prior year Examination questions and Solutions Pamphlets or Problem Books from:

Professor Walter E Mientka, AMC Executive Director
Department of Mathematics and Statistics
University of Nebraska
Lincoln, NE 68588-0322 USA

1. (Answer: 146)

Let $a = x + y$ and $b = xy$, and note that the given equations imply

$$a + b = 71 \qquad \text{and} \qquad ab = 880.$$

Solving simultaneously, one finds that $\{a, b\} = \{16, 55\}$; i.e., either

$$x + y = 55 \qquad \text{and} \qquad xy = 16 \tag{1}$$

or

$$x + y = 16 \qquad \text{and} \qquad xy = 55. \tag{2}$$

It is easy to check that (1) has no solution in integers while in (2) we have $\{x, y\} = \{5, 11\}$. Consequently, $x^2 + y^2 = 5^2 + 11^2 = 146$.

2. (Answer: 840)

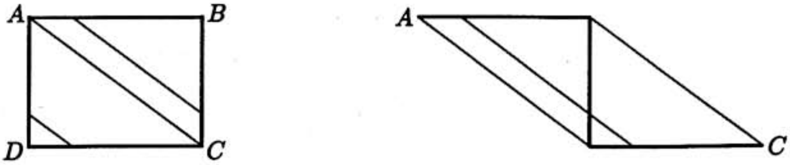
By symmetry, the sum of the lengths of the 335 segments is equal to

$$AC + 2\sum_{k=1}^{167} P_kQ_k.$$

For $1 \leq k \leq 167$ we have $P_kB = AB(1 - k/168)$ and $BQ_k = BC(1 - k/168)$. It follows that $\triangle P_kBQ_k \sim \triangle ABC$, so $P_kQ_k = AC(1 - k/168)$. Thus the sum of the 335 segments is

$$AC\left(1 + 2\sum_{k=1}^{167}\left(1 - \frac{k}{168}\right)\right) = 5\left(1 + \frac{2}{168}\sum_{j=1}^{167} j\right) = 5\left(1 + \frac{2}{168}\frac{167 \cdot 168}{2}\right) = 840.$$

Alternate Solution. Cut the rectangle along diagonal $\overline{AC}$, then reposition $\triangle ABC$ so vertices B and C of $\triangle ABC$ are coincident with vertices A and D, respectively, of $\triangle ADC$. The resulting figure is a parallelogram. In doing this cutting and repositioning we see that, except for diagonal $\overline{AC}$ which has length 5, all of the other 334 segments can be grouped into pairs whose lengths sum to 5. (See figure.) Hence the desired sum is $5 + 167 \cdot 5 = 840$.



3. (Answer: 166)

For $1 \le k \le 1000$,

$$\frac{A_k}{A_{k-1}} = \frac{\frac{1000!}{k!(1000-k)!}(0.2)^k}{\frac{1000!}{(k-1)!(1001-k)!}(0.2)^{k-1}} = \frac{1001-k}{k}(0.2).$$

This ratio never equals 1, and exceeds 1 if and only if $1001 - k > 5k$. This last inequality is true just for $k \le 166$. Thus we have

$$A_0 < A_1 < \ldots < A_{166}$$

while

$$A_{166} > A_{167} > \ldots > A_{1000}.$$

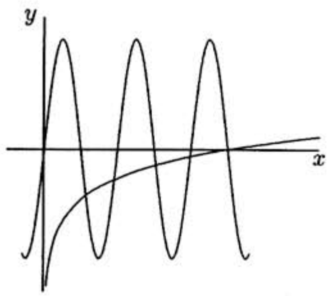
Hence A_k is largest for $k = 166$.

4. (Answer: 159)

Since $|\sin\theta| \le 1$ for all real θ, we need only consider those values of x for which

$$\left|\frac{1}{5}\log_2 x\right| \le 1.$$

This inequality is satisfied by all values of x between $1/32$ and 32, inclusive. We first consider $\frac{1}{32} \le x < 1$. For such x we have $-1 \le \frac{1}{5}\log_2 x < 0$, while $\sin 5\pi x \le 0$ only for $\frac{1}{5} \le x \le \frac{2}{5}$ and $\frac{3}{5} \le x \le \frac{4}{5}$. It follows that the graphs of $y = \frac{1}{5}\log_2 x$ and $y = \sin 5\pi x$ meet at 4 points for $\frac{1}{32} \le x < 1$. (See accompanying figure.) When $1 < x \le 32$, we have $0 < \frac{1}{5}\log_2 x \le 1$ while $\sin 5\pi x \ge 0$ only for $\frac{2k}{5} \le x \le \frac{2k+1}{5}$ $(k = 3, 4, \ldots, 79)$. The graphs of the two functions intersect at 2 points on each of these 77 intervals, giving 154 points of intersection for $1 < x \le 32$. Since both functions take on the value 0 when $x = 1$, we have a total of $4 + 154 + 1 = 159$ solutions to the equation.



5. (Answer: 128)

For a fraction to be in lowest terms, its numerator and denominator must be relatively prime. Thus any prime factor that occurs in the numerator cannot occur in the denominator, and vice-versa. There are eight prime factors of 20!, namely 2, 3, 5, 7, 11, 13, 17, and 19. For each of these prime factors, one must decide only whether it occurs in the numerator or in the denominator. These eight decisions can be made in a total of $2^8 = 256$ ways. However, not all of the 256 resulting fractions will be less than 1. Indeed, they can be grouped into 128 pairs of reciprocals, each of which will have exactly one fraction less than 1. Thus the number of rational numbers with the desired property is 128.

6. (Answer: 743)

The given sum has 73 terms, each of which equals either $\lfloor r \rfloor$ or $\lfloor r \rfloor + 1$. This is because $19/100, 20/100, \ldots, 91/100$ are all less than 1. In order for the sum to be 546, it is necessary that $\lfloor r \rfloor$ be 7, because $73 \cdot 7 < 546 < 73 \cdot 8$. Now suppose that $\lfloor r + \frac{k}{100} \rfloor = 7$ for $19 \le k \le m$ and $\lfloor r + \frac{k}{100} \rfloor = 8$ for $m + 1 \le k \le 91$. Then

$$7(m - 18) + 8(91 - m) = 546,$$

giving $m = 56$. Thus $\lfloor r + \frac{56}{100} \rfloor = 7$ but $\lfloor r + \frac{57}{100} \rfloor = 8$. It follows that $7.43 \le r < 7.44$, and hence that $\lfloor 100r \rfloor = 743$.

7. (Answer: 383)

The fraction on the right side of the equation can be simplified to the form

$$\frac{ax + b}{cx + d},$$

for some real numbers a, b, c, and d. It follows that the given equation is quadratic, and hence has at most 2 solutions. Next, observe that any solution to

$$x = \sqrt{19} + \frac{91}{x} \qquad (*)$$

is also a solution to the original equation. This can be seen by repeatedly replacing each occurrence of x in the right side of $(*)$ by $\sqrt{19} + \dfrac{91}{x}$ until the equation in the problem results. Equation $(*)$ has two solutions,

$$x = \frac{\sqrt{19} + \sqrt{383}}{2} \qquad \text{and} \qquad x = \frac{\sqrt{19} - \sqrt{383}}{2},$$

so these must be the roots of the equation given in the problem. The sum of the absolute values of these roots is $A = \sqrt{383}$, and $A^2 = 383$.

8. (Answer: 010)

Suppose $x^2 + ax + 6a = 0$ has integer roots m and n, with $m \le n$. Since

$$x^2 + ax + 6a = (x - m)(x - n) = x^2 - (m + n)x + mn,$$

we must have $a = -(m + n)$ and $6a = mn$. This implies that a must be an integer and that $-6(m+n) = mn$. This last equation is equivalent to $mn+6m+6n+36 = 36$, or

$$(m + 6)(n + 6) = 36.$$

It is not hard to see that the only integer solutions with $m \le n$ are the ten pairs $(-42, -7)$, $(-24, -8)$, $(-18, -9)$, $(-15, -10)$, $(-12, -12)$, $(-5, 30)$, $(-4, 12)$, $(-3, 6)$, $(-2, 3)$, $(0, 0)$. The corresponding values of $a = -(m + n)$ are 49, 32, 27, 25, 24, -25 -8, -3, -1, and 0. Thus there are 10 values of a for which $x^2 + ax + 6a = 0$ has integer roots.

9. (Answer: 044)

Since $\sec^2 x - \tan^2 x = 1$, we see that $\sec x - \tan x = 1/p$, where p stands for $22/7$. This leads to

$$2\sec x = p + \frac{1}{p} \qquad \text{and} \qquad 2\tan x = p - \frac{1}{p},$$

and then to

$$\cos x = \frac{2p}{p^2 + 1} \qquad \text{and} \qquad \sin x = \frac{p^2 - 1}{p^2 + 1}.$$

It is now an easy matter to show that

$$\csc x + \cot x = \frac{p + 1}{p - 1} = \frac{29}{15}.$$

Alternate solution. We apply the half-angle identities

$$\tan(x/2) = \csc x - \cot x \qquad \text{and} \qquad \cot(x/2) = \csc x + \cot x.$$

Since

$$\frac{22}{7} = \sec x + \tan x = \csc\left(\frac{\pi}{2} + x\right) - \cot\left(\frac{\pi}{2} + x\right) = \tan\left(\frac{1}{2}\left(\frac{\pi}{2} + x\right)\right) = \tan\left(\frac{\pi}{4} + \frac{x}{2}\right),$$

we have

$$\begin{aligned}\frac{m}{n} = \csc x + \cot x = \cot\frac{x}{2} &= \tan\left(\frac{\pi}{2} - \frac{x}{2}\right) = \tan\left(\frac{3\pi}{4} - \left(\frac{\pi}{4} + \frac{x}{2}\right)\right)\\ &= \tan\left(\frac{3\pi}{4} - \arctan\frac{22}{7}\right) = \frac{-1 - \frac{22}{7}}{1 - \frac{22}{7}} = \frac{29}{15}.\end{aligned}$$

10. (Answer: 532)

Let S_a, the three–letter string received when aaa is transmitted, be $x_1x_2x_3$ and let S_b be $y_1y_2y_3$, where each of the x_k, y_k is an a or a b. It will be convenient to introduce the symbol $\prec$ to denote that one string of letters precedes another in alphabetical order. (Thus, if S_1 and S_2 are two strings of letters, then $S_1 \prec S_2$ is to be read "S_1 precedes S_2 alphabetically.") We will find the probability that $S_a \prec S_b$. Since the reception of any one letter is independent of that of any of the other letters, we have

$$\begin{aligned}\text{Prob}(S_a \prec S_b) &= \text{Prob}(x_1x_2x_3 \prec y_1y_2y_3)\\ &= \text{Prob}(x_1 \prec y_1) + \text{Prob}(x_1 = y_1 \text{ and } \mathrm{x}_2 \prec \mathrm{y}_2)\\ &\qquad + \text{Prob}(x_1 = y_1 \text{ and } \mathrm{x}_2 = \mathrm{y}_2 \text{ and } \mathrm{x}_3 \prec \mathrm{y}_3)\\ &= \text{Prob}(x_1 \prec y_1) + \text{Prob}(x_1 = y_1)\cdot\text{Prob}(x_2 \prec y_2)\\ &\qquad + \text{Prob}(x_1 = y_1)\cdot\text{Prob}(x_2 = y_2)\cdot\text{Prob}(x_3 \prec y_3). \qquad (*)\end{aligned}$$

Now $x_1 \prec y_1$ is true if and only if $x_1 = a$ and $y_1 = b$; that is, if and only if these leading letters were received correctly. Since for each letter there is a $\frac{2}{3}$ probability that it was received correctly, we conclude that

$$\text{Prob}(x_1 \prec y_1) = \frac{2}{3}\cdot\frac{2}{3} = \frac{4}{9}.$$

Similarly, $\text{Prob}(x_2 \prec y_2) = \text{Prob}(x_3 \prec y_3) = \frac{4}{9}$. The relation $x_1 = y_1$ is true if and only if one of these letters was received correctly and the other was received incorrectly. Thus

$$\text{Prob}(x_1 = y_1) \;=\; \text{Prob}(x_1 = y_1 = a) + \text{Prob}(x_1 = y_1 = b) \;=\; \frac{2}{3}\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{2}{3} = \frac{4}{9}.$$

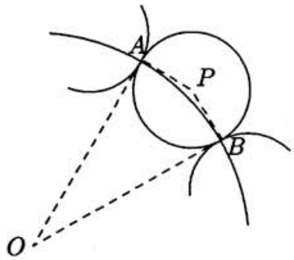
Identical reasoning shows that $\text{Prob}(x_2 = y_2) = \frac{4}{9}$ also. Substituting these probabilities in $(*)$ we have

$$\text{Prob}(S_a \prec S_b) = \frac{4}{9} + \left(\frac{4}{9}\right)^2 + \left(\frac{4}{9}\right)^3 = \frac{532}{729}.$$

The desired numerator is 532.

11. (Answer: 135)

Since the 12 disks cover C and each of the disks is tangent to its two neighbors, C must pass through the 12 points of tangency. The accompanying figure shows one of the covering disks, arcs of the two adjacent disks, and part of C. Let A and B be the points of tangency. By symmetry, the lines mutually tangent to adjacent disks must all pass through the center, O, of C. Let P be the center of the disk shown. Then $\angle PBO$ is a right angle, $\angle BOA = \frac{1}{12}2\pi = \frac{\pi}{6}$ and $\angle POB = \frac{\pi}{12}$. Thus the radius of each of the twelve disks is



$$PB = BO\tan(\angle POB) = \tan\frac{\pi}{12}$$

$$= \tan\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\tan\frac{\pi}{3} - \tan\frac{\pi}{4}}{1 + \tan\frac{\pi}{3}\tan\frac{\pi}{4}} = \frac{\sqrt{3}-1}{1+\sqrt{3}} = 2 - \sqrt{3}.$$

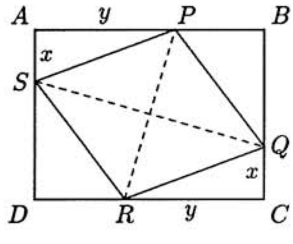
Hence each of the disks has area $\pi(2-\sqrt{3})^2 = \pi(7-4\sqrt{3})$. The sum of the areas of the twelve disks is

$$12\pi(7-4\sqrt{3}) = \pi(84-48\sqrt{3}),$$

and we have $a + b + c = 84 + 48 + 3 = 135$.

12. (Answer: 677)

Let x and y stand for QC and RC respectively, and note that these are also the lengths of $\overline{SA}$ and $\overline{PA}$ as well. The diagonals of a rhombus bisect each other at right angles, hence $PQRS$ and its diagonals divide the rectangle into eight right triangles. Six of these triangles have side lengths of 15, 20, and 25, while the other two have sides of length x, y, and 25. Summing the areas of these eight pieces, we find that



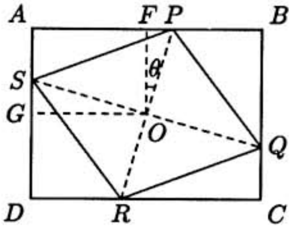
$$6(150) + 2\left(\frac{1}{2}xy\right) = \text{Area(ABCD)}$$
$$= (20+x)(15+y),$$

which leads to $3x+4y = 120$. Combining this with $x^2+y^2 = 625$ leads to the quadratic equation

$$5x^2 - 144x + 880 = 0.$$

Factoring gives $(5x-44)(x-20) = 0$. Note that x cannot be 20 since this would imply $BC = 40$, which is inconsistent with $PR = 30$. Hence $x = 44/5$, $y = 117/5$, and the perimeter of rectangle $ABCD$ is $2(15+20+x+y) = 672/5$.

Alternate Solution. Let x and y stand for AS and AP respectively, and let O denote the intersection of $\overline{PR}$ and $\overline{QS}$. Let F and G, respectively, be the feet of the perpendiculars from O to $\overline{AB}$ and $\overline{AD}$. Then $\angle FOP = \angle GOS$. Let θ be the measure of each of these angles. We then have



$$\cos\theta = \frac{(y+15)/2}{20} = \frac{(x+20)/2}{15}, \quad (*)$$

from which we obtain $3y - 4x = 35$. The equation $3x + 4y = 120$ can be obtained as in the previous solution. Solving the linear equations

$$\begin{aligned} 3x + 4y &= 120 \\ -4x + 3y &= 35 \end{aligned}$$

simultaneously gives $x = 44/5$ and $y = 117/5$. The perimeter of $ABCD$ is thus $2(15 + 20 + x + y) = 672/5$.

Note. From equation $(*)$ it follows that all rectangles that circumscribe a given rhombus have the same shape.

13. (Answer: 990)

Let R and B, respectively, denote the numbers of red and blue socks in the drawer. Because the probability of obtaining a non-matching pair is $1/2$, we have

$$\frac{RB}{\binom{R+B}{2}} = \frac{1}{2}.$$

This leads to $(R+B)(R+B-1) = 4RB$, which can be written as $(R-B)^2 = R+B$. This shows that the total number of socks in the drawer is a perfect square. Let $n = R - B$, so $n^2 = R + B$. Then $R = (n^2 + n)/2$. Since $R + B \leq 1991$, we must have $|n| \leq \sqrt{1991} < 45$. We then see that the largest possible value of R occurs when $n = 44$, and this value of R is 990.

14. (Answer: 384)

Label the remaining four vertices C, D, E, and F, in the natural order. Draw diagonals $\overline{AC}$, $\overline{AD}$, and $\overline{AE}$. and let their lengths be x, y, and z respectively. Draw $\overline{BD}$, $\overline{BE}$, and $\overline{DF}$ and note that $BD = z$, $BE = y$, and $DF = z$, since chords of congruent arcs are congruent. Next we apply Ptolemy's Theorem: *For a quadrilateral inscribed in a circle, the product of the lengths of the diagonals is equal to the sum of the products of the lengths of opposite sides.* Using Ptolemy's Theorem on quadrilaterals $ABCD$, $ABDE$, and $ADEF$ respectively, we obtain

$$\begin{aligned} xz &= 31 \cdot 81 + 81y && (1) \\ y^2 &= 31 \cdot 81 + z^2 \\ z^2 &= 81^2 + 81y. && (2) \end{aligned}$$

Thus $y^2 - 31 \cdot 81 = 81^2 + 81y$, implying

$$y^2 - 81y - 81 \cdot 112 = (y + 63)(y - 144) = 0.$$

Since y cannot be -63, we must have $y = 144$. Substituting in (2) and then in (1) we obtain $z = 135$ and $x = 105$. The sum of the three diagonals is then $x + y + z = 384$.

Alternate Solution. Let R be the radius of the circle and $2x$ the measure of one of the central angles subtended by a side of length 81. Since there are five such sides in the hexagon, we must have $x < 36°$. We now have

$$81 = 2R\sin x \qquad \text{and} \qquad 31 = 2R\sin(\pi - 5x) = 2R\sin 5x, \tag{1}$$

and the sum of the lengths of the three diagonals from A is

$$2R(\sin 2x + \sin 3x + \sin 4x). \tag{2}$$

We will use (1) to find the value of $\sin x$ and then use this to evaluate the sum in (2). By DeMoivre's formula,

$$\begin{aligned} \sin 5x = \operatorname{Im}(\cos 5x + i\sin 5x) &= \operatorname{Im}\left[(\cos x + i\sin x)^5\right] \\ &= 5\cos^4 x \sin x - 10\cos^2 x \sin^3 x + \sin^5 x \\ &= \sin x\left[5(1 - \sin^2 x)^2 - 10\sin^2 x(1 - \sin^2 x) + \sin^4 x\right] \\ &= \sin x(16\sin^4 x - 20\sin^2 x + 5). \end{aligned}$$

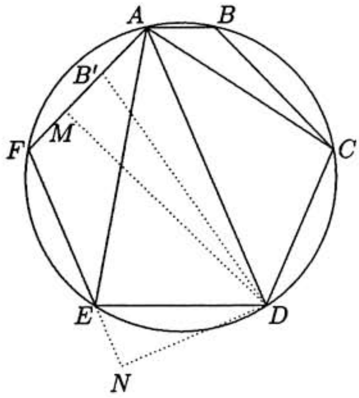
Using (1) we obtain

$$\frac{31}{81} = \frac{\sin 5x}{\sin x} = 16\sin^4 x - 20\sin^2 x + 5 = \left(4\sin^2 x - \frac{5}{2}\right)^2 - \frac{5}{4},$$

from which $\sin x = \pm\sqrt{11}/6$ or $\pm\sqrt{34}/6$. Since $x < 36°$ and $\sin x$ must be positive, we conclude that $\sin x = \sqrt{11}/6$ and $\cos x = 5/6$. We can now evaluate the sum in (2):

$$\begin{aligned}
&2R(\sin 2x + \sin 3x + \sin 4x)\\
&\quad= \frac{81}{\sin x}\left[2\sin x\cos x + (3\sin x - 4\sin^3 x) + 4\sin x\cos x(1-2\sin^2 x)\right]\\
&\quad= 81\left[2\cos x + (3 - 4\sin^2 x) + 4\cos x(1 - 2\sin^2 x)\right]\\
&\quad= 81\left[\frac{5}{3} + \left(3 - \frac{11}{9}\right) + \frac{10}{3}\left(1 - \frac{11}{18}\right)\right]\\
&\quad= 135 + 243 - 99 + 270 - 165\\
&\quad= 384.
\end{aligned}$$

Alternate Solution Label the remaining vertices C, D, E, F. We shall find and make use of a few pairs of similar right triangles to calculate AD, AE, and AC, in this order.

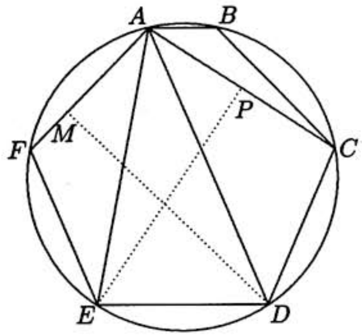


Mark point B' on $\overline{AF}$ so that $AB = AB'$. Since $\angle BAD$ and $\angle FAD$ intercept congruent arcs, these angles are congruent, and it follows that $\triangle BAD$ and $\triangle B'AD$ are congruent. Thus $BD = B'D$. Since $BD = FD$ is also true, we have $B'D = FD$. If we now let M denote the midpoint of $\overline{B'F}$ we see that $\triangle AMD$ is a right triangle with leg $\overline{AM}$ of length 56. Now drop a perpendicular from D to the extension of $\overline{FE}$ and let N be the foot of this perpendicular. Since trapezoid $ADEF$ is isosceles, we deduce that $\angle FAD = \angle NED$, and hence that $\triangle AMD \sim \triangle END$. Thus

$$\frac{56}{AD} = \frac{AM}{AD} = \frac{EN}{ED} = \frac{\frac{AD-81}{2}}{81},$$

from which $AD^2 - 81AD - 2\cdot 56\cdot 81 = 0$. The solutions to this quadratic are -63 and 144, hence $AD = 144$.

The preceding argument has also shown that $EN = 63/2$. The Pythagorean theorem now gives us $DN = 45\sqrt{11}/2$, and a second application, to $\triangle FND$, gives $FD = 135$. This is also the length of $\overline{AE}$.



To find AC, consider isosceles $\triangle AEC$. Let P be the midpoint of $\overline{AC}$, so that $\triangle PAE$ is a right triangle. This triangle is similar to $\triangle MAD$ since $\angle PAE = \angle MAD$. We thus obtain

$$\frac{AP}{AM} = \frac{AE}{AD},$$

from which

$$AP = (AM)(AE)/AD = \frac{105}{2},$$

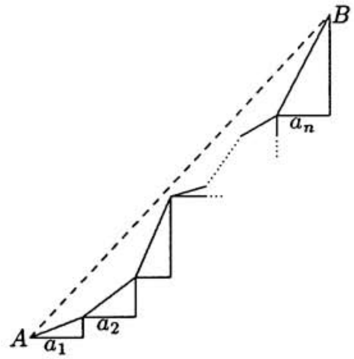
and $AC = 2AP = 105$. The required sum is therefore $AC + AD + AE = 384$.

15. (Answer: 012)

We interpret each term

$$t_k = \sqrt{(2k-1)^2 + a_k^2}$$

as the length of the hypotenuse of a right triangle with legs of length $2k-1$ and a_k. Put the triangles together in a "staircase" arrangement as shown in the diagram, and let A and B be the initial and terminal points of the broken path formed by the hypotenuses. The distance from A to B is



$$\sqrt{\left(\sum_{k=1}^{n} a_k\right)^2 + \left(\sum_{k=1}^{n}(2k-1)\right)^2} = \sqrt{17^2 + n^4},$$

while the sum $\sum_{k=1}^{n} t_k$ is the length of the path from A to B formed by the hypotenuses of the triangles. It follows immediately that $\sum_{k=1}^{n} t_k \geq \sqrt{17^2 + n^4}$, and that equality is obtained by choosing the a_k so that the broken path is actually a straight line. Thus $S_n = \sqrt{17^2 + n^4}$ is the minimum possible value of the given sum. When S_n is an integer, the equation $17^2 = S_n^2 - n^4 = (S_n - n^2)(S_n + n^2)$ implies that

$$S_n + n^2 = 17^2$$

and

$$S_n - n^2 = 1.$$

Solving this system yields $S_n = 145$ and $n = 12$.