AMERICAN MATHEMATICS COMPETITIONS

AIME SOLUTIONS PAMPHLET
FOR STUDENTS AND TEACHERS

8th ANNUAL
AMERICAN INVITATIONAL MATHEMATICS EXAMINATION
(AIME)
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This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs. geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers will share these solutions with their students, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution. *Remember that reproduction of these solutions is prohibited by copyright.*

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1. (Answer: 528)
Between 1 and 500, there are $\lfloor\sqrt{500}\rfloor = 22$ perfect squares and $\lfloor\sqrt[3]{500}\rfloor = 7$ perfect cubes. Among these integers there are $\lfloor\sqrt[6]{500}\rfloor = 2$ of them (1 and 64) that are counted twice. Thus there are $22 + 7 - 2 = 27$ integers between 1 and 500 that are not in the sequence. To get the 500^{th} number, we must append 27 integers to the list $2, 3, 5, \ldots, 500$ of 473 non-squares and non-cubes. Since we cannot use 512, the last number will be 528.

2. (Answer: 828)
We split 52 into two parts to obtain squares in each set of parentheses:

$$\begin{aligned}
&(52+6\sqrt{43})^{3/2}-(52-6\sqrt{43})^{3/2}\\
&\quad=(43+6\sqrt{43}+9)^{3/2}-(43-6\sqrt{43}+9)^{3/2}\\
&\quad=\left[(\sqrt{43}+3)^2\right]^{3/2}-\left[(\sqrt{43}-3)^2\right]^{3/2}\\
&\quad=(\sqrt{43}+3)^3-(\sqrt{43}-3)^3\\
&\quad=(43\sqrt{43}+3\cdot3\cdot43+3\cdot3^2\sqrt{43}+3^3)\\
&\qquad\qquad-(43\sqrt{43}-3\cdot3\cdot43+3\cdot3^2\sqrt{43}-3^3)\\
&\quad=828.
\end{aligned}$$

Alternate Solution. Let $\alpha = (52+6\sqrt{43})^{1/2}$ and $\beta = (52-6\sqrt{43})^{1/2}$. We wish to find $\alpha^3 - \beta^3 = (\alpha-\beta)(\alpha^2+\alpha\beta+\beta^2)$. Now

$$\alpha^2+\beta^2 = 104 \qquad \text{and} \qquad \alpha\beta = (52^2 - 36\cdot 43)^{1/2} = (1156)^{1/2} = 34.$$

Thus $(\alpha-\beta)^2 = \alpha^2 - 2\alpha\beta + \beta^2 = 104 - 68 = 36$, so $\alpha - \beta = 6$ and $\alpha^3 - \beta^3 = 6(104+34) = 828$.

3. (Answer: 117)
In a regular n-gon, each interior angle has radian measure $(n-2)\pi/n$. The information in the problem says

$$\frac{59}{58} = \left(\frac{r-2}{r}\pi\right)\bigg/\left(\frac{s-2}{s}\pi\right) = \frac{rs-2s}{rs-2r}. \qquad (*)$$

Solving for r gives

$$r = \frac{116s}{118-s}.$$

Since r must be positive, we must have $s \le 117$. Indeed, if $s = 117$ then we find $r = 116\cdot 117$ and equation $(*)$ will be satisfied.

4. (Answer: 013)
Let $x^2 - 10x = y$. The equation in the problem then becomes

$$\frac{1}{y-29} + \frac{1}{y-45} - \frac{2}{y-69} = 0,$$

from which

$$\frac{1}{y-29} - \frac{1}{y-69} = \frac{1}{y-69} - \frac{1}{y-45},$$

and

$$\frac{-40}{(y-29)(y-69)} = \frac{24}{(y-45)(y-69)}$$

follows. This equation has $y = 39$ as its only solution. We then note that $x^2 - 10x = 39$ is satisfied by the positive number 13.

5. (Answer: 432)
Suppose the prime factorization of n has the form

$$n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k},$$

where $p_1, p_2, \ldots, p_k$ are the distinct prime divisors of n and $r_1, r_2, \ldots, r_k$ are positive integers. Then the number of divisors of n is given by

$$(r_1 + 1)(r_2 + 1) \cdots (r_k + 1).$$

Since this last product must be $75 = 3 \cdot 5 \cdot 5$, we see that n can have at most three distinct prime factors. To ensure that n is divisible by 75 and that the n we obtain is minimal, the prime factors must belong to the set $\{2, 3, 5\}$, with the factor 3 occurring at least once and the factor 5 occurring at least twice. Thus

$$n = 2^{r_1} 3^{r_2} 5^{r_3}$$

with

$$(r_1 + 1)(r_2 + 1)(r_3 + 1) = 75 \qquad r_2 \geq 1, r_3 \geq 2.$$

It is not hard to write down the ordered triples (r_1, r_2, r_3) that satisfy the above conditions:

$$(4,4,2) \qquad (4,2,4) \qquad (2,4,4) \qquad (0,4,14)$$
$$(0,14,4) \qquad (0,2,24) \qquad (0,24,2).$$

Among the above ordered triples, the minimum value for n occurs when $r_1 = r_2 = 4$ and $r_3 = 2$. Thus our answer is $n/75 = 2^4 3^3 = 432$.

6. (Answer: 840)
Let

$$X = \text{the number of fish in the lake on May 1,}$$
$$Y = \text{the number of fish in the lake on September 1.}$$

From the data in the problem we find that $Y = .75X + .40Y$, and that the number of tagged fish in the lake on September 1 is $.75(60) = 45$. Thus, assuming the tagged fish are fairly represented in the September 1 sample, we have

$$\frac{3}{70} = \frac{45}{Y}.$$

Hence $Y = 1050$ and $X = .60Y/.75 = 840$.

7. (Answer: 089)
Extend $\overline{PR}$ through R to T, where T is selected so that $PQ = PT$. Since $PQ = 25$ and $PR = 15$, the point T has coordinates

$$\begin{aligned} P + \frac{25}{15}(R - P) &= (-8,5) + \frac{25}{15}[(1,-7) - (-8,5)] \\ &= (-8,5) + \frac{5}{3}(9,-12) \\ &= (7,-15). \end{aligned}$$

Now $\angle P$ in $\triangle PQR$ and $\angle P$ in $\triangle PQT$ have the same bisector. Since $\triangle PQT$ is isosceles, with $PQ = PT$, this bisector intersects $\overline{QT}$ at its midpoint, $(-4,-17)$. Thus the slope of the bisector is $-\frac{11}{2}$ and its equation can be written in the form $11x + 2y + 78 = 0$. Hence $a + c = 11 + 78 = 89$.

Alternate Solution. Consider the vectors $\overrightarrow{PQ} = -7\vec{\imath} - 24\vec{\jmath}$ and $\overrightarrow{PR} = 9\vec{\imath} - 12\vec{\jmath}$. Let $\vec{v} = a\vec{\imath} + b\vec{\jmath}$ be a vector parallel to the bisector of $\angle P$. Then the angle between vectors $\overrightarrow{PQ}$ and $\vec{v}$ is equal to the angle between vectors $\vec{v}$ and $\overrightarrow{PR}$. Let ϕ be the measure of each of these angles. Then

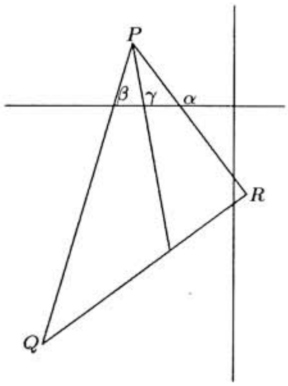
$$\frac{\vec{v} \cdot \overrightarrow{PQ}}{\|\vec{v}\|\,\|\overrightarrow{PQ}\|} = \cos\phi = \frac{\vec{v} \cdot \overrightarrow{PR}}{\|\vec{v}\|\,\|\overrightarrow{PR}\|}$$

giving

$$\frac{-7a - 24b}{25} = \frac{9a - 12b}{15},$$

which simplifies to $11a + 2b = 0$. Hence $b = -\frac{11}{2}a$ and it follows that the bisector of $\angle P$ has slope $-\frac{11}{2}$. The equation of the bisector can be written as $11x + 2y + 78 = 0$, so $a + c = 11 + 78 = 89$.

Alternate Solution. Let α, β, γ, respectively, be the angles that $\overline{PR}$, $\overline{PQ}$ and the bisector of $\angle P$ make with the x-axis. (These angles are measured counterclockwise from the x-axis.) Let m be the slope of the bisector of $\angle P$. Then the slope of $\overline{PR}$ is $\tan\alpha = -\frac{4}{3}$, the slope of $\overline{PQ}$ is $\tan\beta = \frac{24}{7}$, and the slope of the bisector of $\angle P$ is $\tan\gamma = m$. Since $\alpha - \gamma = \frac{1}{2}\angle P = \gamma - \beta$, we have $\tan(\alpha-\gamma) = \tan(\gamma-\beta)$. Using the formula for the tangent of the difference of two angles gives



$$\frac{\tan\alpha - \tan\gamma}{1+\tan\alpha\tan\gamma} = \frac{\tan\gamma - \tan\beta}{1+\tan\gamma\tan\beta},$$

which leads to

$$\frac{-\frac{4}{3} - m}{1 - \frac{4}{3}m} = \frac{m - \frac{24}{7}}{1 + \frac{24}{7}m}.$$

The last equation has solutions $m = -\frac{11}{2}$ and $m = \frac{2}{11}$. The solution $-\frac{11}{2}$ is the slope of the internal bisector of $\angle P$. (Some other line has slope $\frac{2}{11}$. Which one?) We then find that the equation of the bisector can be written in the form $11x + 2y + 78 = 0$, and $a + c = 11 + 78 = 89$.

8. (Answer: 560)

Consider the eight shots that must be fired to break the eight targets. Of the eight, any subset of three shots may be the shots used to break the targets in the first column (but once these three shots are chosen the rules of the match determine the order in which the targets in the first column will be broken by these shots.) This set of shots for the first column may be chosen in $\binom{8}{3}$ ways. From the remaining five shots, the three used to break the targets in the other column of three may be chosen in $\binom{5}{3}$ ways, while the remaining two shots will be used to break the remaining two targets. Combining, we find that the number of orders in which the targets can be broken is

$$\binom{8}{3}\binom{5}{3}\binom{2}{2} = \binom{8}{3,3,2} = \frac{8!}{3!3!2!} = 560.$$

9. (Answer: 073)
There are 2^n possible sequences of length n that can be formed from the letters **T** and **H**. Let $A(n)$ be the number of these sequences in which there are no adjacent occurrences of **H**. The values $A(1) = 2$, $A(2) = 3$ and $A(3) = 5$ may be found by simply listing all possible outcomes for tosses of 1, 2 and 3 coins respectively. For higher values of n, we may find $A(n)$ by using the recursion relation $A(n+2) = A(n+1) + A(n)$, which holds for any positive integer n. This recursion relation is true because $A(n+2)$ counts two distinct types of sequences: those with no consecutive **H**'s that end with **T** (there are $A(n+1)$ of them) and those with no consecutive **H**'s that end with **TH** (there are $A(n)$ of these).

It follows that the values $A(n)$ are Fibonacci numbers, so $A(10) = 144$. Hence the probability of tossing a coin ten times and never having heads occur on consecutive tosses is $144/1024 = 9/64$ and $i + j = 73$.

10. (Answer: 144)
First observe that if $z \in A$ and $w \in B$, then

$$(zw)^{144} = (z^{18})^8 (w^{48})^3 = 1.$$

This shows that the set C is contained in the set of 144^{th} roots of unity. Next we show that any 144^{th} root of unity is in C, thereby showing that C has 144 elements. Let x be a 144^{th} root of unity. Then there is an integer k with

$$x = \cos\left(\frac{2\pi}{144}k\right) + i\sin\left(\frac{2\pi}{144}k\right) = \operatorname{cis}\left(\frac{2\pi}{144}k\right) = \left[\operatorname{cis}\left(\frac{2\pi}{144}\right)\right]^k,$$

where the last equality follows by an application of DeMoivre's formula. We next express the greatest common divisor of 18 and 48 as $6 = 3 \cdot 18 - 48$ and use this in the following:

$$\operatorname{cis}\left(\frac{2\pi}{144}\right) = \operatorname{cis}\left(\frac{2\pi}{864}\cdot 6\right) = \operatorname{cis}\left(\frac{2\pi}{864}(3\cdot 18 - 48)\right) = \operatorname{cis}\left(\frac{2\pi}{48}3\right)\operatorname{cis}\left(\frac{2\pi}{18}(-1)\right).$$

By another application of DeMoivre's formula, we now have

$$x = \left[\operatorname{cis}\left(\frac{2\pi}{48}3\right)\operatorname{cis}\left(\frac{2\pi}{18}(-1)\right)\right]^k = \operatorname{cis}\left(\frac{2\pi}{48}3k\right)\operatorname{cis}\left(\frac{2\pi}{18}(-k)\right),$$

which shows that x is a product of elements from A and B. Hence the set of 144^{th} roots of unity is a subset of C. We may conclude that C is the set of 144^{th} roots of unity, so C has 144 elements.

11. (Answer: 023)
If $n!$ can be expressed as the product of $n-3$ consecutive integers, then there is a positive integer k such that

$$n! = (n+k)(n+k-1)\cdots(k+4) = \frac{(n+k)!}{(k+3)!}.$$

We can express this last relation as

$$\frac{(n+k)!/n!}{(k+3)!} = 1$$

and expand to get

$$\frac{n+k}{k+3}\cdot\frac{n+k-1}{k+2}\cdots\frac{n+2}{5}\cdot\frac{n+1}{4!} = 1.$$

If $n > 23$, then the k factors on the left of the previous equation all exceed 1 and the equation cannot be true. On the other hand, $n = 23$ and $k = 1$ is an obvious solution to (1) and shows that $n = 23$ is the answer to the problem.

Note. The above argument can be generalized to prove the following result: *Let $r \geq 2$ be an integer. The largest integer n for which $n!$ can be written as the product of $n-r$ consecutive positive integers is $n = (r+1)! - 1$.*

12. (Answer: 720)
Position the 12-gon in the Cartesian plane with its center at the origin and one vertex at $(12, 0)$. Compute the sum, S, of the lengths of the eleven segments emanating from this vertex. The coordinates of the other vertices are given by $(12\cos kx, 12\sin kx)$ where $x = 30°$ and $k = 1, 2, \ldots, 11$. The length of the segment joining $(12, 0)$ to $(12\cos kx, 12\sin kx)$ is

$$12\sqrt{(\cos kx - 1)^2 + (\sin kx)^2} = 12\sqrt{2 - 2\cos kx} = 24\sin\frac{kx}{2}.$$

Thus the sum of the lengths of the 11 segments from $(12, 0)$ is

$$S = 24(\sin 15° + \sin 30° + \cdots + \sin 150° + \sin 165°).$$

Since $\sin t = \sin(180° - t)$ we may write

$$S = 48(\sin 15° + \sin 30° + \sin 45° + \sin 60° + \sin 75°) + 24\sin 90°.$$

Now

$$\begin{aligned}\sin 15^\circ + \sin 75^\circ &= \sin(45^\circ - 30^\circ) + \sin(45^\circ + 30^\circ)\\ &= 2\sin 45^\circ \cos 30^\circ\\ &= \frac{1}{2}\sqrt{6}.\end{aligned}$$

Thus

$$\begin{aligned}S &= 48\Big(\frac{1}{2}\sqrt{6} + \frac{1}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\Big) + 24\\ &= 48 + 24\sqrt{2} + 24\sqrt{3} + 24\sqrt{6}.\end{aligned}$$

The same value, S, occurs if we add the lengths of all segments emanating from any other vertex of the 12-gon. Since each segment is counted at two vertices (its endpoints) the total length of all such segments is

$$\frac{1}{2}(12S) = 288 + 144\sqrt{2} + 144\sqrt{3} + 144\sqrt{6}.$$

Hence $a + b + c + d = 288 + 144 + 144 + 144 = 5 \cdot 144 = 720$.

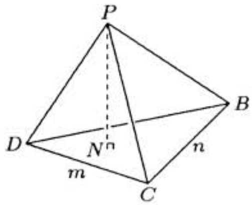
13. (Answer: 184)

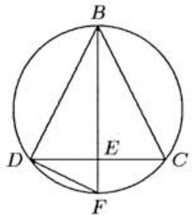
Note that 9^k has one more digit than 9^{k-1}, except in the case when 9^k starts with a 9. In the latter case, long division shows that 9^{k-1} starts with a 1 and has the same number of digits as 9^k. Therefore, when the powers of 9 from 9^0 to 9^{4000} are computed there are 3816 increases in the number of digits. Thus there must be $4000 - 3816 = 184$ instances when computing 9^k from 9^{k-1} ($1 \le k \le 4000$) does not increase the number of digits. Since $9^0 = 1$ does not have leading digit 9 we can conclude that 9^k ($1 \le k \le 4000$) has a leading digit of 9 exactly when there is no increase in the number of digits in computing 9^k from 9^{k-1}. It follows that 184 of the numbers must start with the digit 9.

Note. We did not need to know that the leading digit of 9^{4000} is 9, but it was important to note that the leading digit of 9^0 is *not* 9.

14. (Answer: 594)

Let m and n denote CD and BC, respectively. Three of the faces are lettered PBC, PCD, and PDB (see diagram). Let N be the point where the altitude from P meets BCD. We first show that N is the circumcenter of $\triangle BCD$. To see this, note that $\triangle PNB$, $\triangle PNC$ and $\triangle PND$ are congruent by the hypotenuse-leg criterion. It follows that $\overline{BN}$, $\overline{CN}$ and $\overline{DN}$ all have the same length r; this r is the radius (and hence N is the center) of the circle that circumscribes $\triangle BCD$. We will find the value of r and use it to find PN. From B draw a diameter of the circumcircle. Let the other end of this diameter be F and let E be the point where the diameter meets $\overline{CD}$. Then $\angle BFD \cong \angle ECB$ since both angles subtend the same arc on the circumcircle. Hence the two right triangles $\triangle BFD$ and $\triangle BCE$ are similar, implying $BF/BD = BC/BE$. Since $BF = 2r$, the last equation gives





$$r = \frac{n^2}{2(BE)} = \frac{n^2}{\sqrt{4n^2 - m^2}}.$$

Now using $PB = \frac{1}{2}\sqrt{m^2 + n^2}$ we can find PN, the altitude of the pyramid, by the Pythagorean theorem:

$$PN^2 = PB^2 - BN^2 = \frac{m^2 + n^2}{4} - \frac{n^4}{4n^2 - m^2}.$$

Hence

$$PN = \frac{m}{2}\sqrt{\frac{3n^2 - m^2}{4n^2 - m^2}}.$$

Thus, the volume of pyramid $PBCD$ is

$$\frac{1}{3}(PN)(\text{Area}(\triangle BCD)) = \frac{1}{3} \cdot \frac{m}{2}\sqrt{\frac{3n^2 - m^2}{4n^2 - m^2}} \cdot \frac{m}{2}\sqrt{n^2 - \frac{m^2}{4}} = \frac{m^2}{24}\sqrt{3n^2 - m^2}.$$

Since $m^2 = 432$ and $n^2 = 507$, the volume is 594.

15. (Answer: 020)
For $n = 1$ and $n = 2$, the identity

$$(1) \qquad (ax^{n+1} + by^{n+1})(x+y) - (ax^n + by^n)xy = ax^{n+2} + by^{n+2}$$

yields the equations

$$7(x+y) - 3xy = 16 \qquad \text{and} \qquad 16(x+y) - 7xy = 42.$$

Solving these last two equations simultaneously, one finds that

$$(2) \qquad x + y = -14 \qquad \text{and} \qquad xy = -38.$$

Applying (1) with $n = 3$ then gives

$$ax^5 + by^5 = (42)(-14) - (16)(-38) = -588 + 608 = 20.$$

Note: From (2) we can solve for x and y. We obtain $x = -7 \pm \sqrt{87}$ and $y = -7 \mp \sqrt{87}$, from which $a = \frac{49}{76} \pm \frac{457}{6612}\sqrt{87}$ and $b = \frac{49}{76} \mp \frac{457}{6612}\sqrt{87}$.