AMERICAN MATHEMATICS COMPETITIONS

AIME SOLUTIONS PAMPHLET
FOR STUDENTS AND TEACHERS

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AMERICAN INVITATIONAL MATHEMATICS EXAMINATION
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This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs. geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution. *Remember that reproduction of these solutions is prohibited by copyright.*

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1. (Answer: 728)
We first count those integers of the desired type with 4 or 6 as the thousands digit. In this case, the thousands digit can be chosen in 2 ways, and then the units digit (0, 2, 4, 6, or 8) can be chosen in 4 ways. There are then 8 choices for the hundreds digit and then 7 for the tens digit. Thus there are $2 \cdot 4 \cdot 8 \cdot 7 = 448$ integers of the type we seek with 4 or 6 as the thousands digit. Similarly, if the thousands digit is 5, we have $1 \cdot 5 \cdot 8 \cdot 7 = 280$ even integers with four different digits. Thus we have a total of $448 + 280 = 728$ integers.

2. (Answer: 580)
Consider a Cartesian coordinate system with origin at the candidate's starting point, positive x-axis pointing east, and positive y-axis pointing north. At the end of the 40^{th} day the candidate is at the point

$$\begin{aligned}
&\left(\frac{1^2-3^2+5^2-7^2+\cdots+37^2-39^2}{2}, \frac{2^2-4^2+6^2-8^2+\cdots+38^2-40^2}{2}\right)\\
=&\left(\frac{(1-3)(1+3)+(5-7)(5+7)+\cdots+(37-39)(37+39)}{2},\right.\\
&\qquad\left.\frac{(2-4)(2+4)+(6-8)(6+8)+\cdots+(38-40)(38+40)}{2}\right)\\
=&(-4-12-20-\cdots-76, -6-14-22-\cdots-78)\\
=&\left(10\frac{-4-76}{2}, 10\frac{-6-78}{2}\right)\\
=&(-400, -420).
\end{aligned}$$

Thus the candidate's distance from his starting point is $\sqrt{400^2+420^2} = 580$.

3. (Answer: 943)
Let F be the total number of fish caught during the festival and C be the total number of contestants. Then $C-(9+5+7) = C-21$ contestants each caught 3 or more fish, and these contestants caught a total of $F-(0\cdot 9+1\cdot 5+2\cdot 7) = F-19$ fish. Hence

$$\frac{F-19}{C-21} = 6. \qquad (1)$$

Similarly, $C-(5+2+1) = C-8$ contestants each caught 12 or fewer fish, and these contestants caught a total of $F-(5\cdot 13+2\cdot 14+1\cdot 15) = F-108$ fish. Thus

$$\frac{F-108}{C-8} = 5. \qquad (2)$$

Solving (1) and (2) simultaneously, we find $C = 175$ and $F = 943$.

4. (Answer: 870)
Since $a + d = b + c$, we may take $(a, b, c, d) = (a, a + x, a + y, a + x + y)$, where x and y are integers with $0 < x < y$. Then

$$93 = bc - ad = (a + x)(a + y) - a(a + x + y) = xy,$$

from which either $(x, y) = (1, 93)$ or $(x, y) = (3, 31)$. In the first case,

$$(a, b, c, d) = (a, a + 1, a + 93, a + 94)$$

is in the desired range for $a = 1, 2, \ldots, 405$. In the second case,

$$(a, b, c, d) = (a, a + 3, a + 31, a + 34)$$

is in the desired range for $a = 1, 2, \ldots, 465$. These two sets of four-tuples are disjoint, so a total of $405 + 465 = 870$ four-tuples of integers (a, b, c, d) satisfy the given conditions.

5. (Answer: 763)
For positive integers n, we have

$$\begin{aligned} P_n(x) &= P_{n-1}(x - n) \\ &= P_{n-2}(x - n - (n - 1)) \\ &= P_{n-3}(x - n - (n - 1) - (n - 2)) \\ &\;\;\vdots \\ &= P_0(x - n - (n - 1) - \cdots - 2 - 1), \end{aligned}$$

from which

$$P_n(x) = P_0\left(x - \frac{1}{2}n(n + 1)\right).$$

Hence

$$P_{20}(x) = P_0\left(x - \frac{1}{2}20 \cdot 21\right) = P_0(x-210) = (x-210)^3+313(x-210)^2-77(x-210)-8.$$

The coefficient of x in this polynomial is

$$3(210)^2 - 313 \cdot 2 \cdot 210 - 77 = 210(630 - 626) - 77 = 763.$$

6. (Answer: 495)
The sum of nine consecutive integers is 9 times the fifth number, the sum of ten consecutive integers is 5 times the sum of the fifth and sixth numbers, and the sum of eleven consecutive integers is 11 times the sixth number. Thus any positive integer that can be written as a sum of nine, ten, and eleven consecutive positive integers must be a multiple of 9, 5, and 11. The smallest such number is 495. It is readily verified that

$$\begin{aligned} 495 &= 51 + 52 + \cdots + 59 \\ &= 45 + 46 + \cdots + 54 \\ &= 40 + 41 + \cdots + 50. \end{aligned}$$

7. (Answer: 005)
Since we may rotate the brick before we attempt to place it in the box, we may assume that $a_1 < a_2 < a_3$ and that $b_1 < b_2 < b_3$. The brick will then fit in the box if and only if $a_1 < b_1$, $a_2 < b_2$, and $a_3 < b_3$. Because each selection of 6 dimensions is equally likely, there is no loss of generality in assuming the brick and box dimensions are selected from the set $\{1, 2, 3, 4, 5, 6\}$. There are $\binom{6}{3} = 20$ ways to select the dimensions of a brick-box pair from $\{1, 2, 3, 4, 5, 6\}$.
If the brick does fit inside the box, then we must have $a_1 = 1$ and $b_3 = 6$. In addition, we must have $b_2 > b_1 > a_1 = 1$ and $b_2 > a_2$, so $6 > b_2 > 3$.
If $b_2 = 4$, then $a_3 = 5$. Taking b_1 to be either 2 or 3 will result in a pair of dimensions for which the brick fits in the box.
If $b_2 = 5$, then taking b_1 to be 2, 3, or 4 will result in a box that can hold the brick.
Thus there are 5 ways to select the two sets of dimensions from $\{1, 2, 3, 4, 5, 6\}$ so that the brick fits inside the box. It follows that the probability that the brick will fit inside the box is $5/20 = 1/4$. The sum of the numerator and denominator is 5.

8. (Answer: 365)
In order that $A \cup B = S$, for each element s of S exactly one of the following three statements is true:

$s \in A$ and $s \notin B$ $\quad$ $s \notin A$ and $s \in B$ $\quad$ $s \in A$ and $s \in B$.

Hence if S has n elements, there are 3^n ways to choose the sets A and B. Except for pairs with $A = B$, this total counts each pair of sets twice. Since $A \cup B = S$ with $A = B$ occurs if and only if $A = B = S$, the number of pairs of subsets of S whose union is S is

$$\frac{3^n - 1}{2} + 1,$$

which is 365 when $n = 6$.

Alternate Solution. Let S be the set with n elements and A and B be two subsets whose union is S. If $|A| = k$, then B must contain the $n - k$ elements of S not in A. There are $\binom{n}{k}$ such sets A. Each of the k elements in A may or may not be in B, so for each set A there are 2^k sets B that can be paired with A. Note that

$$\sum_{k=0}^{n} \binom{n}{k} 2^k = (1+2)^n = 3^n$$

counts each pair of sets twice, except the pair $A = B = S$, which occurs once. Hence the number of pairs of subsets whose union is S is $(3^n + 1)/2$. When $n = 6$, this gives 365.

9. (Answer: 118)

Let A be the point labeled 1. For any positive integer n we can find the point labeled by n by counting $1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1)$ points around the circle in the clockwise direction, with the count starting at A. It follows that two positive integers l and m will label the same point if and only if $\frac{1}{2}l(l+1)$ and $\frac{1}{2}m(m+1)$ have the same remainder when divided by 2000. Thus, if k is a positive integer that labels the same point as 1993, then

$$2\left(\frac{1993(1993+1)}{2} - \frac{k(k+1)}{2}\right) = (1993 - k)(1994 + k)$$

must be a multiple of $4000 = 2^5 5^3$. It is clear that $k = 1993$ satisfies these conditions; we need to see if there is a positive integer $k < 1993$ that also satisfies these conditions and, if any exist, find the smallest such integer. Since $1993 - k$ and $1994 + k$ are of different parity and cannot both be multiples of 5, one of these integers must be a multiple of 125 and one must be a multiple of 32. If $k < 1993$, then $1994 + k < 32 \cdot 125 = 4000$, so exactly one of $1993 - k$ and $1994 + k$ is a multiple of 125 and the other is a multiple of 32. We consider these two cases.

Case 1. $125 \mid (1993 - k)$ and $32 \mid (1994 + k)$

Because $1993 = 15 \cdot 125 + 118$ and $1994 = 62 \cdot 32 + 10 = 63 \cdot 32 - 22$, it follows that $k - 118$ and $k - 22$ are divisible by 125 and 32, respectively. In other words, $k = 118 + 125r$ and $k = 22 + 32s$ for non-negative integers r and s. It is now evident that $k \geq 118$ and that $k = 118$ arises from $r = 0$ and $s = 3$.

Case 2. $125 \mid (1994 + k)$ and $32 \mid (1993 - k)$

Because $1994 = 15 \cdot 125 + 119$ and $1993 = 62 \cdot 32 + 9$, it follows that $k + 119$ and $k - 9$ are divisible by 125 and 32 respectively. Thus $k = 125r - 119$ and $k = 32s + 9$ for non-negative integers r and s. From this we obtain $125r = 128 + 32s$, so r is a multiple of 32. Thus for some integer t we have $k = 125 \cdot 32t - 119$. It follows that any positive integer k satisfying this case is greater than 1993.

Hence 118 is the smallest positive integer that labels the same point as 1993.

Note. In analyzing the above cases, we are solving systems of congruences. In Case 1 the system is

$$k \equiv 118 \pmod{125}$$
$$k \equiv 22 \pmod{32}.$$

Since 32 and 125 are relatively prime, it follows from the Chinese Remainder Theorem that a solution to the system exists, and that all solutions are congruent modulo $125 \cdot 32 = 4000$.

10. (Answer: 250)
Since $F - E + V = 2$, and $F = 32$ it follows that

$$E = V + 30.$$

Since $T + P$ faces meet at each vertex, there are $T + P$ edges that meet at each vertex. Hence $2E = V(T + P)$, from which $V(T + P) = 2(V + 30)$ and

$$V(T + P - 2) = 60. \qquad (1)$$

Each triangular face has three vertices, so the product VT counts each triangular face 3 times. Thus the total number of triangular faces is $VT/3$. Similarly, the total number of pentagonal faces is $VP/5$. Because every face is a triangle or a pentagon,

$$V\left(\frac{T}{3} + \frac{P}{5}\right) = 32. \qquad (2)$$

Combining (1) and (2) we have

$$60V\left(\frac{T}{3} + \frac{P}{5}\right) = 32V(T + P - 2),$$

from which

$$3T + 5P = 16.$$

The only non-negative integer solution of this equation is $T = P = 2$. From (1) we find $V = 30$, so $100P + 10T + V = 250$.

Note. The polyhedron described above is called an icosidodecahedron.

11. (Answer: 093)

For any game the probability that the first player wins the game is

$$\frac{1}{2}+\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^5+\cdots=\frac{\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2}=\frac{2}{3}.$$

Hence the probability that the second player wins the game is $1-\frac{2}{3}=\frac{1}{3}$. Now let P_k denote the probability that Alfred wins the k^{th} game. Then $P_1=\frac{2}{3}$ and for $k\geq 2$ we have

$$P_k=\frac{1}{3}P_{k-1}+\frac{2}{3}(1-P_{k-1})=\frac{2}{3}-\frac{1}{3}P_{k-1},$$

from which

$$P_k-\frac{1}{2}=-\frac{1}{3}\left(P_{k-1}-\frac{1}{2}\right).$$

It follows that

$$P_k=\frac{1}{2}+\frac{(-1)^{k-1}}{3^{k-1}}\left(P_1-\frac{1}{2}\right)=\frac{1}{2}+\frac{(-1)^{k-1}}{2\cdot 3^k}.$$

When $k=6$, this probability is $364/729$. Then $m+n=1093$ and the last three digits are 093.

Note. The probability P_1 that the person going first wins the game can be computed without using geometric series. Note that after each player tosses a tail, the game essentially starts anew. Hence

$$P_1=\frac{1}{2}+\left(\frac{1}{2}\right)^2 P_1,$$

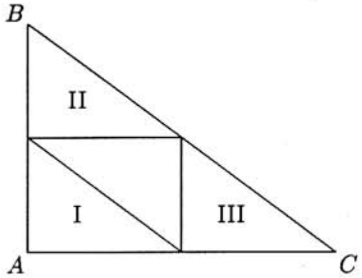
from which we find $P_1=2/3$.

12. (Answer: 344)
First note that, since P_1 is inside $\triangle ABC$, all subsequent points P_k will also be inside the triangle. Furthermore, as will be shown below, once any subsequent P_k is given, then P_1 is uniquely determined. Suppose that $P_k = (x_k, y_k)$ is known. Since P_k is inside $\triangle ABC$, we have

$$0 < x_k < 560, \qquad 0 < y_k < 420, \qquad 0 < 420x_k + 560y_k < 420 \cdot 560.$$

If A is rolled, then

$$(x_{k+1}, y_{k+1}) = P_{k+1} = \frac{1}{2}P_k = \left(\frac{x_k}{2}, \frac{y_k}{2}\right),$$



so the range of possible positions of P_{k+1} is limited to the original triangle contracted by a factor of 1/2 (region I in the diagram). Hence if A is rolled, then P_{k+1} is in the interior of region I, and we may conclude that

$$420x_{k+1} + 560y_{k+1} < \frac{1}{2} \cdot 420 \cdot 560.$$

Similarly, if B is rolled, then P_{k+1} is in the interior of region II, so $y_{k+1} > 210$. If C is rolled, then P_{k+1} is in the interior of region III, so $x_{k+1} > 280$. Thus, for $k \ge 2$, P_k must lie in one of the regions I,II,III, and its predecessor is uniquely determined. For example if $P_k = (x_k, y_k)$ lies in region II, then P_k must be the midpoint of $\overline{BP_{k-1}}$. It follows that $P_{k-1} = 2P_k - B = (2x_k, 2y_k - 420)$. We can now construct a "predecessor function" as follows: if $k \ge 2$ and $P_k = (x_k, y_k)$, then

$$P_{k-1} = \begin{cases} (2x_k, 2y_k - 420) & \text{if } y_k > 210 \\ (2x_k - 560, 2y_k) & \text{if } x_k > 280 \\ (2x_k, 2y_k) & \text{if } 420x_k + 560y_k < \frac{1}{2}420 \cdot 560 \end{cases}$$

It is now easy to trace $P_7 = (14, 92)$ back to P_1:

$$P_7 = (14,92) \implies P_6 = (28,184) \implies P_5 = (56,368) \implies P_4 = (112,316) \implies$$
$$P_3 = (224,212) \implies P_2 = (448,4) \implies P_1 = (336,8).$$

We then see that $k + m = 336 + 8 = 344$.

Query. What is the set of all points P_7 for which there exists a corresponding P_1 inside the triangle?

Note. The above analysis shows that the point $P_7 = (14, 92)$ is generated from $P_1 = (336, 8)$ by the sequence of die outcomes C, B, B, B, A, A.

If a fair die is used in generating points $P_2, P_3, \cdots, P_N$ for some large value of N, the graph will almost always resemble the Sierpinski triangle, one of the better known fractals.

13. (Answer: 163)

Let A and C, respectively, be Kenny's and Jenny's positions at the instant when the building first blocks their line of sight, and let B and D be their positions when they can first see each other again. Let P be the point where $\overline{AC}$ extended meets $\overline{BD}$ extended. These two segments are tangent to the building and $\overline{AC} \perp \overline{AB}$. Let O be the center of the building, F the point at which $\overline{AC}$ is tangent to the building, and t the time in seconds that passes as Kenny walks from A to B. Then $AB = 3t$, $CD = t$, $CF = FA = 100$, and $OF = 50$. Since $\triangle PAB \sim \triangle PCD$, we see that

$$3 = \frac{AB}{CD} = \frac{PA}{PC} = \frac{200 + PC}{PC},$$

so $PC = 100$. Let θ be the measure of $\angle FPO$. Since $\overrightarrow{PO}$ is the bisector of $\angle BPA$, it follows that $\angle BPA$ has measure 2θ. Thus

$$t = CD = PC \tan 2\theta = 100 \frac{2\tan\theta}{1 - \tan^2\theta}.$$

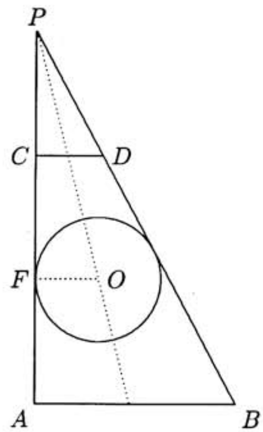
Using

$$\tan\theta = \frac{OF}{PF} = \frac{50}{200} = \frac{1}{4},$$

we compute

$$t = 100 \frac{2 \cdot \frac{1}{4}}{1 - \left(\frac{1}{4}\right)^2} = \frac{160}{3}.$$

The sum of the numerator and denominator is 163.



Alternate Solution. Let Kenny walk to the right along the x-axis, Jenny to the right along the line $y = 200$, and let the building be centered at $(50, 100)$. Then Kenny and Jenny lose sight of each other as they (simultaneously) cross the y-axis. If we assume this happens at time 0, then at time t, Kenny's position will be $(3t, 0)$ and Jenny's position will be $(t, 200)$. An equation for the line determined by Kenny's and Jenny's positions at time t is

$$100x + ty - 300t = 0.$$

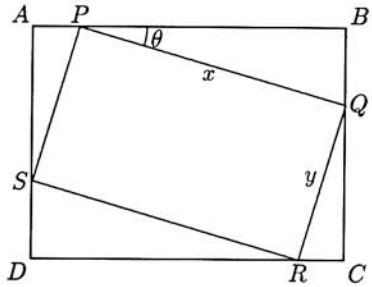
We wish to find the time $t > 0$ when the distance from the center of the building to this line is 50. Using the formula for the distance from a point to a line, we have

$$\frac{|100 \cdot 50 + t \cdot 100 - 300t|}{\sqrt{100^2 + t^2}} = 50,$$

which reduces to $15t^2 - 800t = 0$. This will be the case when $t = 0$ or $t = 160/3$.

14. (Answer: 448)

Let $ABCD$ be the outer rectangle, with $AB = 8$. Let $PQRS$ be an inscribed rectangle, with P, Q, R, S on $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ respectively. Note that angles QPB, RQC, SRD, and PSA are congruent; let θ denote their common measure. Let $x = PQ$ and $y = QR$. Without loss of generality, we may assume that the figure has been labeled so that $y < x$ and $\theta < 45°$. (Why?) Then



$$BA = BP + PA = x\cos\theta + y\sin\theta = 8$$
$$BC = BQ + QC = x\sin\theta + y\cos\theta = 6,$$

and these equations can be solved simultaneously to give

$$x = \frac{8\cos\theta - 6\sin\theta}{\cos^2\theta - \sin^2\theta} \qquad \text{and} \qquad y = \frac{6\cos\theta - 8\sin\theta}{\cos^2\theta - \sin^2\theta}.$$

Thus the perimeter of $PQRS$ is

$$2x + 2y = \frac{28}{\cos\theta + \sin\theta}.$$

Since $\cos\theta + \sin\theta = \sqrt{2}\sin(\theta + 45°)$ and $0 < \theta < 45°$, it follows that the perimeter of $PQRS$ is a decreasing function of θ. It will be shown below that BQ is an increasing function of θ, and that $PQRS$ is stuck if and only if $3 < BQ$. Therefore, we seek the perimeter of the rectangle that results when Q is the midpoint of $\overline{BC}$ (and S is the midpoint of $\overline{DA}$). Because Q and S are midpoints of opposite sides of $ABCD$, rectangle $PQRS$ has area 24 and diagonal 8, yielding the equations

$$xy = 24$$
$$x^2 + y^2 = 64.$$

Combining these, we find that $(x+y)^2 = 64+2\cdot 24 = 112$, so $2(x+y) = 2\sqrt{112} = \sqrt{448}$ is the smallest perimeter for an unstuck inscribed rectangle.

To complete the demonstration, let O be the intersection of $\overline{AC}$ and $\overline{BD}$, and consider circles centered at O and intersecting all four sides of $ABCD$. These circles all have diameters between 8 and 10. Except for the extreme cases, each such circle intersects $ABCD$ at eight points, P, P', Q, Q', R, R', S, S', given in cyclical order so that P and P' are on $\overline{AB}$, Q and Q' are on $\overline{BC}$, etc. Note that $PQRS$ and $PQ'RS'$ are inscribed rectangles.

Rectangle $PQRS$ is unstuck (Figure 1), because its vertices can be moved along the arcs PS', QP', RQ', and SR', which lie inside $ABCD$. Note that $\angle QOP' = 2\theta$. As the diameter decreases, both 2θ and BQ increase, because P' and Q drift away from B. This shows that BQ is an increasing function of θ.

Rectangle $PQ'RS'$ is stuck (Figure 2) when the diameter exceeds 8 (and $3 < BQ'$). This is because arcs $Q'Q$ and RR' are outside $ABCD$, so Q' and R are free to move only along arc $Q'R$, which is impossible.

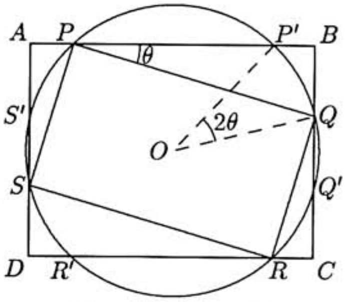


Figure 1 (Unstuck)

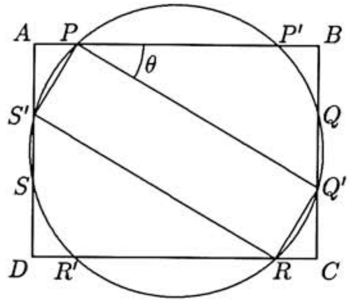
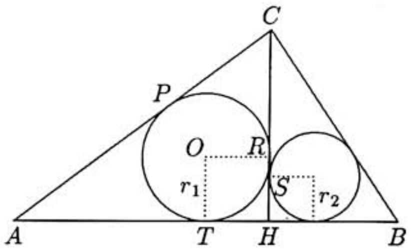


Figure 2 (Stuck)

Query. We have shown that $x+y$ is a decreasing function of θ. Are x and y themselves decreasing functions of θ?

15. (Answer: 997)

Let $a = BC$, $b = AC$, $c = AB$, $h = CH$, $p = AH$, and $q = BH$. Let O be the center of the circle inscribed in $\triangle AHC$, let r_1 be the radius of this circle, and let T and P, respectively, be the points where this circle is tangent to $\overline{AB}$ and $\overline{AC}$. Since $\angle CHA$ is a right angle, we have $\overline{OR} \perp \overline{OT}$, and hence $RH = OT = r_1$. Similarly, $SH = r_2$, where r_2 is the radius of the circle inscribed in $\triangle CHB$. Thus



$$RS = |RH - SH| = |r_1 - r_2|.$$

Next note that

$$b = AC = AP + CP = AT + CR = (p - r_1) + (h - r_1),$$

from which $r_1 = (p + h - b)/2$. Similarly, $r_2 = (q + h - a)/2$.
Thus

$$RS = |r_1 - r_2| = \left|\frac{p+h-b}{2} - \frac{q+h-a}{2}\right| = \frac{1}{2}|(p-q) + (a-b)|. \qquad (*)$$

By the Pythagorean Theorem, $a^2 - q^2 = h^2 = b^2 - p^2$, so $p^2 - q^2 = b^2 - a^2$. From this we have

$$p - q = \frac{(b+a)(b-a)}{p+q} = \frac{(b+a)(b-a)}{c}.$$

Substituting this last expression into (∗) gives

$$RS = \frac{1}{2}\left|\frac{(b+a)(b-a)}{c} + (a-b)\right| = \frac{|b-a|}{2c}|a+b-c|.$$

With $a = 1993$, $b = 1994$, and $c = 1995$, we find

$$RS = \frac{1}{2 \cdot 1995}1992 = \frac{332}{665},$$

so $m + n = 332 + 665 = 997$.