AMERICAN MATHEMATICS COMPETITIONS

13th ANNUAL
AMERICAN INVITATIONAL MATHEMATICS EXAMINATION (AIME)

THURSDAY, MARCH 23, 1995

Sponsored by

Mathematical Association of America
Society of Actuaries Mu Alpha Theta
National Council of Teachers of Mathematics
Casualty Actuarial Society American Statistical Association
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1. DO NOT OPEN THIS BOOKLET UNTIL TOLD TO DO SO BY YOUR PROCTOR.
2. This is a 15-question, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers; i.e., there is neither partial credit nor a penalty for wrong answers.
3. No aids other than scratch paper, graph paper, ruler, compass, and protractor are permitted. In particular, **calculators are not permitted.**
4. Please print the following:

Name: ______________________________
Last First Middle initial

Home address: ______________________________
Street Address

____
City State or Province Zip or Postcode

____
Home Phone including Area Code Gender (M or F) Your age

____
Full Name of School Grade Level (e.g.,11)

5. Citizenship Status: USA Citizen_____ *Permanent Resident_____ Other_____

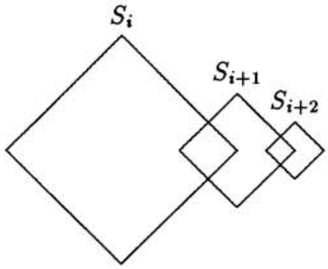
   If other, explain: ______________________________

   *Permanent Resident means someone seeking citizenship and currently possessing a U.S.A. Immigration "green card".
6. A combination of the AIME and AHSME scores is used to determine eligibility for participation in the U.S.A. Mathematical Olympiad (USAMO). The USAMO will be given on THURSDAY, April 27, 1995. Please check one box:

   If I qualify for the USAMO, I agree to take it. YES ☐ NO ☐

   (*Your school must also agree to administer the USAMO before you can take it.*)
7. Record all your answers, and certain other information, on the AIME answer form. Your Examination Manager will instruct you how to complete the form. Only the answer form and this cover will be collected from you.

1. Square S_1 is 1×1. For $i \geq 1$, the lengths of the sides of square S_{i+1} are half the lengths of the sides of square S_i, two adjacent sides of square S_i are perpendicular bisectors of two adjacent sides of square S_{i+1}, and the other two sides of square S_{i+1} are the perpendicular bisectors of two adjacent sides of square S_{i+2}. The total area enclosed by at least one of S_1, S_2, S_3, S_4, S_5 can be written in the form m/n, where m and n are relatively prime positive integers. Find $m - n$.

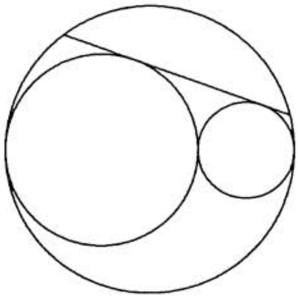


2. Find the last three digits of the product of the positive roots of

$$\sqrt{1995}\, x^{\log_{1995} x} = x^2.$$

3. Starting at (0,0), an object moves in the coordinate plane via a sequence of steps, each of length one. Each step is left, right, up, or down, all four equally likely. Let p be the probability that the object reaches $(2,2)$ in six or fewer steps. Given that p can be written in the form m/n, where m and n are relatively prime positive integers, find $m + n$.

4. Circles of radius 3 and 6 are externally tangent to each other and are internally tangent to a circle of radius 9. The circle of radius 9 has a chord that is a common external tangent of the other two circles. Find the square of the length of this chord.



5. For certain real values of a, b, c, and d, the equation $x^4 + ax^3 + bx^2 + cx + d = 0$ has four non-real roots. The product of two of these roots is $13 + i$ and the sum of the other two roots is $3 + 4i$, where $i = \sqrt{-1}$. Find b.

6. Let $n = 2^{31}3^{19}$. How many positive integer divisors of n^2 are less than n but do not divide n?

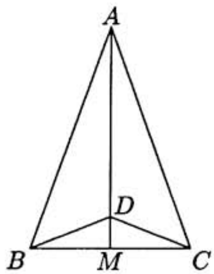
7. Given that $(1 + \sin t)(1 + \cos t) = 5/4$ and

$$(1 - \sin t)(1 - \cos t) = \frac{m}{n} - \sqrt{k},$$

where k, m, and n are positive integers with m and n relatively prime, find $k + m + n$.

8. For how many ordered pairs of positive integers (x, y), with $y < x \le 100$, are both $\frac{x}{y}$ and $\frac{x+1}{y+1}$ integers?

9. Triangle ABC is isosceles, with $AB = AC$ and altitude $AM = 11$. Suppose that there is a point D on $\overline{AM}$ with $AD = 10$ and $\angle BDC = 3\angle BAC$. Then the perimeter of $\triangle ABC$ may be written in the form $a + \sqrt{b}$, where a and b are integers. Find $a + b$.



10. What is the largest positive integer that is not the sum of a positive integral multiple of 42 and a positive composite integer?

11. A right rectangular prism P (i.e., a rectangular parallelepiped) has sides of integral length a, b, c, with $a \le b \le c$. A plane parallel to one of the faces of P cuts P into two prisms, one of which is similar to P, and both of which have nonzero volume. Given that $b = 1995$, for how many ordered triples (a, b, c) does such a plane exist?

12. Pyramid $OABCD$ has square base $ABCD$, congruent edges $\overline{OA}$, $\overline{OB}$, $\overline{OC}$, and $\overline{OD}$, and $\angle AOB = 45°$. Let θ be the measure of the dihedral angle formed by faces OAB and OBC. Given that $\cos\theta = m + \sqrt{n}$, where m and n are integers, find $m + n$.

13. Let $f(n)$ be the integer closest to $\sqrt[4]{n}$. Find $\displaystyle\sum_{k=1}^{1995} \frac{1}{f(k)}$.

14. In a circle of radius 42, two chords of length 78 intersect at a point whose distance from the center is 18. The two chords divide the interior of the circle into four regions. Two of these regions are bordered by segments of unequal lengths, and the area of either of them can be expressed uniquely in the form $m\pi - n\sqrt{d}$, where m, n, and d are positive integers and d is not divisible by the square of any prime number. Find $m + n + d$.

15. Let p be the probability that, in the process of repeatedly flipping a fair coin, one will encounter a run of 5 heads before one encounters a run of 2 tails. Given that p can be written in the form m/n, where m and n are relatively prime positive integers, find $m + n$.

SOLUTIONS

A 1995 Solutions Pamphlet will be sent to exam managers within a few weeks.

WRITE TO US!

Correspondence about the problems and solutions for this AIME should be addressed to:

Mr. Richard Parris, AIME Chairman
Department of Mathematics
Phillips Exeter Academy, Exeter, NH 03833-2460 USA

Comments about administrative arrangements and orders for any publications listed below should be addressed to:

Prof Walter E Mientka, AMC Executive Director
Department of Mathematics and Statistics
University of Nebraska, Lincoln, NE 68588-0658 USA

1995 USAMO

The USA Mathematical Olympiad is a 5-question, $3\frac{1}{2}$ hour, essay-type examination. The USAMO will be held on THURSDAY, April 27, 1995. Your teacher has more details on who qualifies for the USAMO in the AHSME or AIME Teachers' Manuals. The best way to prepare for the USAMO is to study the exams from previous years and to review the contents of the ARBELOS. Copies may be ordered as indicated below.

PUBLICATIONS

MINIMUM ORDER: $5 (before handling fee), US FUNDS ONLY. Canada and US orders must be prepaid. Orders are mailed 4th class, unless you specify 1st class, in which case add 20% of total order with a $3.00 minimum and a $15.00 maximum. Make checks payable to the American Mathematics Competitions; or give Visa or Mastercard number, expiration date and name on the card.

FOREIGN ORDERS: do **NOT** prepay; an invoice will be sent.

Examinations: Each price is for **one** copy of an exam **and** its solutions for one year. Specify the years you want and how many copies of each. All prices effective to September 1, 1995.

- **AJHSME** (Junior High Exam), 1985-1994, $1 per copy per year.
- **AHSME** 1980-95, $1 per copy per year.
- **AIME** 1983-95, $2 per copy per year.
- USA and International Mathematical **Olympiads** (together), 1976-94 (1995 available in September), $5 per copy per year.
- National **Summary** of Results and Awards, 1980-94, $ 10 per copy per year.

Books (Exams and solutions):

- Contest Problem Book I, AHSMEs 1950-60, $11.50.
- Contest Problem Book II, AHSMEs 1961-65, $11.50.
- Contest Problem Book III, AHSMEs 1966-72, $13.50.
- Contest Problem Book IV, AHSMEs 1973-82, $13.50.
- USA Mathematical Olympiads, 1972-86, $14.50.
- International Mathematical Olympiads, 1959-77, $12.50.
- International Mathematical Olympiads, 1978-85, $11.50.

Journal

- The ARBELOS (short articles and challenging problems); recommended especially for AIME and USAMO qualifiers. Five volumes plus a Geometry volume, $7.00 each.