AIME SOLUTIONS PAMPHLET

FOR STUDENTS AND TEACHERS

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This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs. geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

It is hoped that teachers will find the time to share these solutions with their students, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution.

AMERICAN MATHEMATICS COMPETITIONS

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1. (Answer: 770)

   There are 2^{10} configurations of the ten buttons. We are to exclude those in which none or all of the buttons are depressed, as well as those in which exactly five buttons are depressed. Thus the total number of additional combinations is

$$2^{10} - 1 - 1 - \binom{10}{5} = 1024 - 2 - 252 = 770.$$

2. (169)

   First we observe that, perhaps after a nonrepeating initial segment, the sequence $f_1(11)$, $f_2(11)$, ... is periodic. To see this, it suffices to note that, for $k < 1000$,

$$f_1(k) \le f_1(999) = (9+9+9)^2 = 729 < 1000.$$

   Next we compute $f_n(11)$ for the first few values of n, in the hope that the length of the period is short. This expectation is reasonable since the terms of the sequence are perfect squares, and since there are only 31 perfect squares less than 1000. We find that

$$f_1(11) = (1+1)^2 = 4,$$
$$f_2(11) = f_1(f_1(11)) = f_1(4) = 4^2 = 16,$$
$$f_3(11) = f_1(f_2(11)) = f_1(16) = (1+6)^2 = 49,$$
$$f_4(11) = f_1(f_3(11)) = f_1(49) = (4+9)^2 = 169,$$
$$f_5(11) = f_1(f_4(11)) = f_1(169) = (1+6+9)^2 = 256,$$
$$f_6(11) = f_1(f_5(11)) = f_1(256) = (2+5+6)^2 = 169.$$

   We stop at this point, since $f_n(11)$ depends only on $f_{n-1}(11)$, and hence the numbers 256 and 169 will continue to alternate. More precisely, for $n \ge 4$,

$$f_n(11) = \begin{cases} 169, & \text{if } n \text{ is even,} \\ 256, & \text{if } n \text{ is odd.} \end{cases}$$

   Since 1988 is even, it follows that $f_{1988}(11) = 169$.

3. (027)

Since $\log_8 x = \frac{1}{\log_x 8} = \frac{1}{3\log_x 2} = \frac{1}{3}\log_2 x$ and $\log_8(\log_2 x) = \frac{1}{3}\log_2(\log_2 x)$,

the given equation is equivalent to

(1) $$\log_2(y/3) = (1/3)\log_2 y,$$

where $y = \log_2 x$. From (1), $\log_2(y/3)^3 = \log_2 y$, hence $(y/3)^3 = y$; i.e.,

(2) $$y(y^2 - 27) = 0.$$

Since $y \neq 0$ (for otherwise, neither side of (1) would be defined), it follows from (2) that $y^2 = (\log_2 x)^2 = 27$.

4. (020)

If n is a positive integer, then

(1) $$\sum_{k=1}^{n} |x_k| - |\sum_{k=1}^{n} x_k| \leq \sum_{k=1}^{n} |x_k| < n,$$

since $|\sum_{k=1}^{n} x_k| \geq 0$ and $|x_k| < 1$ for $1 \leq k \leq n$. Since it is given that

(2) $$\sum_{k=1}^{n} |x_k| - |\sum_{k=1}^{n} x_k| = 19,$$

it follows that $19 < n$. Thus the answer is 20 if there is a solution to (2) with $n = 20$. One such solution is

$$x_k = \begin{cases} .95 & \text{if } k \text{ is odd,} \\ -.95 & \text{if } k \text{ is even.} \end{cases}$$

5. (634)

The divisors of 10^{99} are of the form $2^a \cdot 5^b$, where a and b are integers with $0 \leq a \leq 99$ and $0 \leq b \leq 99$. Since there are 100 choices for both a and b, 10^{99} has $100 \cdot 100$ positive integer divisors. Of these, the multiples of $10^{88} = 2^{88} \cdot 5^{88}$ must satisfy the inequalities $88 \leq a \leq 99$ and $88 \leq b \leq 99$. Thus there are 12 choices for both a and b; i.e., $12 \cdot 12$ of the $100 \cdot 100$ divisors of 10^{99} are multiples of 10^{88}. Consequently, the desired probability is $\frac{m}{n} = \frac{12 \cdot 12}{100 \cdot 100} = \frac{9}{625}$ and $m + n = 634$.

6. (142)

Let a and b denote the numbers in two of the squares as shown in the first figure below, and compute the two neighboring entries in terms of them. Then the common difference in the third row is $b - 2a$, while in the fourth row it is $2b - a - 74$. Consequently,

$$2a + 4(b - 2a) = 186 \quad \text{and} \quad a + 2(2b - a - 74) = 103.$$

Solving these equations simultaneously, we find that $a = 13$ and $b = 66$. Therefore, the entries in the third and fourth row, and then in the fourth column may be computed to find that the number in the square marked by the asterisk (*) is 142. For completeness, the second figure shows the rest of the entries as well.

			*	
	74			
2a	b			186
a	2b-74	103		
0				

52	82	112	142	172
39	74	109	144	179
26	66	106	146	186
13	58	103	148	193
0	50	100	150	200

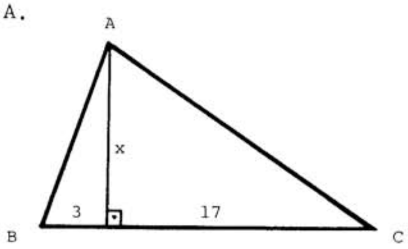
7. (110)

Let x be the length of the altitude from A. Then

$$\tan^{-1}(22/7) = \tan^{-1}(3/x) + \tan^{-1}(17/x).$$

Taking tangents of both sides, and using the formula for $\tan(\alpha + \beta)$, we obtain

$$\frac{22}{7} = \frac{(3/x) + (17/x)}{1 - (3/x)(17/x)},$$



which simplifies to

$$11x^2 - 70x - 561 = (x - 11)(11x + 51) = 0.$$

Since x must be positive, we conclude that $x = 11$, and that the area of ΔABC is $(1/2)(3 + 17)(11) = 110$.

8. (364)

The third property is most useful in the form $f(x,z) = \frac{z}{z-x} f(x,z-x)$, which is valid whenever $z > x$. To obtain this, set $z = y + x$, so that $y = z - x$. Now substitute for y in the original third property. Using this new form of the third property and the second given property of f repeatedly, we obtain

$$\begin{aligned} f(14,52) &= \frac{52}{38} f(14,38) = \frac{52}{38}\frac{38}{24} f(14,24) = \frac{52}{24}\frac{24}{10} f(14,10) \\ &= \frac{26}{5} f(10,14) = \frac{26}{5}\frac{14}{4} f(10,4) = \frac{91}{5} f(4,10) = \frac{91}{5}\frac{10}{6} f(4,6) \\ &= \frac{91}{3}\frac{6}{2} f(4,2) = 91\, f(2,4) = 91\frac{4}{2} f(2,2) = 364, \end{aligned}$$

where the last equality is a consequence of the first given property of f.

Note. The computations may be somewhat simplified if we introduce a new function g by letting $g(x,y) = \frac{1}{xy} f(x,y)$. The student should verify that $f(x,y)$ is the least common multiple of x and y, while $g(x,y)$ is the reciprocal of the greatest common divisor of x and y. In fact, computing with g is like using Euclid's Algorithm in slow motion!

9. (192)

If the cube of an integer ends in 8, then the integer itself must end in 2; i.e., must be of the form $10k + 2$. Therefore,

(1) $$n^3 = (10k+2)^3 = 1000k^3 + 600k^2 + 120k + 8,$$

where the penultimate term, $120k$, determines the penultimate digit of n^3, which must also be 8. In view of this, $12k$ must also end in 8; i.e., k must end in 4 or 9, and hence be of the form $5m + 4$. Thus

(2) $$\begin{aligned} n^3 &= (10(5m+4) + 2)^3 \\ &= 125000m^3 + 315000m^2 + 264600m + 74088. \end{aligned}$$

Since the first two terms on the right of (2) end in 000, while the last term ends in 088, it follows that $264600m$ must end in 800. The smallest m which will ensure this is $m = 3$, implying that $k = 5 \cdot 3 + 4 = 19$, and $n = 10 \cdot 19 + 2 = 192$. (Indeed, $192^3 = 7{,}077{,}888$.)

10. (840)

Let V, E, D and I denote the number of vertices, edges, facial diagonals and interior diagonals of the polyhedron. We will evaluate them in succession, starting with V.

Since at each vertex of the polyhedron there is exactly one of each kind of face, and since the 4 vertices of each of the square faces must be at different vertices of the polyhedron,

$$V = 4 \times \text{number of square faces} = 4 \times 12 = 48.$$

(Note that the same value results if we consider the hexagonal or octagonal faces.)

Since there are 3 edges emanating from each vertex of the polyhedron, and since 3V counts each edge twice,

$$E = \frac{1}{2} \times 3V = 72.$$

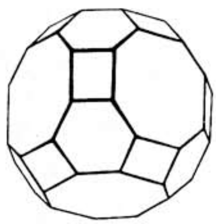
To find D, note that each square face has 2 diagonals, each hexagonal face has 9 diagonals, and each octagonal face has 20 diagonals. Consequently,

$$D = 12 \times 2 + 8 \times 9 + 6 \times 20 = 216.$$

Finally, because the polyhedron is convex, we may find I by subtracting the number of edges and the number of facial diagonals from the total number of ways to connect pairs of vertices for the polyhedron. Thus

$$I = \binom{48}{2} - E - D = 1128 - 72 - 216 = 840.$$

Note. The polyhedron in question is the "great rhombicuboctahedron", also known as the (rhombi)truncated cuboctahedron. It is one of the thirteen semi-regular (Archimedean) solids. See Fejes Tóth's Regular Figures (Pergamon Press, 1964) or Coxeter's Regular Polytopes (Dover, 1973).



11. (163)

Let $y = mx + b$ be a mean line for the complex numbers $w_k = u_k + iv_k$, where u_k and v_k are real, and $k = 1, 2, \ldots, n$. Assume that the complex numbers $z_k = x_k + iy_k$, where x_k and y_k are real, are chosen on the line $y = mx + b$ so that

$$\sum_{k=1}^{n} (z_k - w_k) = 0.$$

Then

$$\sum x_k = \sum u_k, \quad \sum y_k = \sum v_k, \quad \text{and} \quad y_k = mx_k + b \quad (1 \le k \le n),$$

where $\sum$ means summation as k ranges from 1 to n. Consequently,

$$\sum v_k = \sum y_k = \sum (mx_k + b) = m\sum x_k + nb = (\sum u_k)m + nb.$$

Since in our case, $n = 5$, $b = 3$, $\sum u_k = 3$, and $\sum v_k = 504$, it follows that $504 = 3m + 15$ and hence $m = 163$.

Note. We have shown only that $m = 163$ is necessary for $y = mx + 3$ to be a mean line for the given set of points. The reader should find corresponding $z_1, z_2, \ldots, z_5$ to verify the sufficiency of this choice for m.

The definition of mean line given in this problem is not the standard one; i.e., usually it is also required that each of the segments connecting w_k to z_k be perpendicular to the mean line. Can the reader show that the two definitions are equivalent, and that a mean line for a set of complex numbers is nothing more than a line through the centroid of the set?

Query. What if $\sum u_k = 0$?

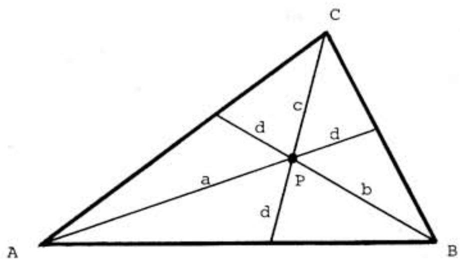
12. (441)

First observe that

(1) $$\frac{d}{d + a} = \frac{\text{area}(\Delta BPC)}{\text{area}(\Delta BAC)},$$

(2) $$\frac{d}{d + b} = \frac{\text{area}(\Delta CPA)}{\text{area}(\Delta CBA)},$$

(3) $$\frac{d}{d + c} = \frac{\text{area}(\Delta APB)}{\text{area}(\Delta ACB)}.$$



Then, since area($\triangle BPC$) + area($\triangle CPA$) + area($\triangle APB$) = area($\triangle ABC$), the sum of (1), (2) and (3) simplifies to

$$\frac{d}{d+a} + \frac{d}{d+b} + \frac{d}{d+c} = 1.$$

Multiplying through by $(d+a)(d+b)(d+c)$, expanding and grouping like terms, we may write the above as

(4) $$2d^3 + (a+b+c)d^2 - abc = 0.$$

From (4), in view of the given data it follows that $abc = 2\cdot 3^3 + 43\cdot 3^2 = 441$.

Note. One such triangle has $a = b = 21$ and $c = 1$. Show that there are infinitely many noncongruent triangles meeting the conditions of the problem. Is it true that if two of the quantities $a+b+c$, abc, d are given, then the third is uniquely determined and can be realized geometrically?

13. (987)

Since the roots of $x^2 - x - 1 = 0$ are $p = \frac{1}{2}(1+\sqrt{5})$ and $q = \frac{1}{2}(1-\sqrt{5})$, these must also be roots of $ax^{17} + bx^{16} + 1 = 0$. Thus

$$ap^{17} + bp^{16} = -1 \qquad \text{and} \qquad aq^{17} + bq^{16} = -1.$$

Multiplying the first of these equations by q^{16}, the second one by p^{16}, and using the fact that $pq = -1$, we find that

$$ap + b = -q^{16} \qquad \text{and} \qquad aq + b = -p^{16}.$$

Thus

(1) $$a = \frac{p^{16} - q^{16}}{p - q} = (p^8 + q^8)(p^4 + q^4)(p^2 + q^2)(p + q).$$

Since

$$p + q = 1,$$
$$p^2 + q^2 = (p + q)^2 - 2pq = 1 + 2 = 3,$$
$$p^4 + q^4 = (p^2 + q^2)^2 - 2(pq)^2 = 9 - 2 = 7,$$
$$p^8 + q^8 = (p^4 + q^4)^2 - 2(pq)^4 = 49 - 2 = 47,$$

substitution into (1) yields

$$a = (47)(7)(3)(1) = 987.$$

Note. Substituting for p and q into (1) gives

$$a = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{16} - \left(\frac{1-\sqrt{5}}{2}\right)^{16}\right],$$

whose right side may be recognized as (the Binet form of) the 16-th Fibonacci number F_{16}. It is easiest to compute F_{16} by iteration, using the recursive definition: $F_1 = F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n > 2$.

Alternate Solution. The other factor is of degree 15 and we write (with slight malice aforethought)

$$(-1 - x + x^2)(-c_0 + c_1 x - \dots + c_{15}x^{15}) = 1 + bx^{16} + ax^{17}.$$

Comparing coefficients:

$$x^0: \quad c_0 = 1,$$
$$x^1: \quad c_0 - c_1 = 0 \quad \Rightarrow \quad c_1 = 1,$$
$$x^2: \quad -c_0 - c_1 + c_2 = 0 \quad \Rightarrow \quad c_2 = 2,$$

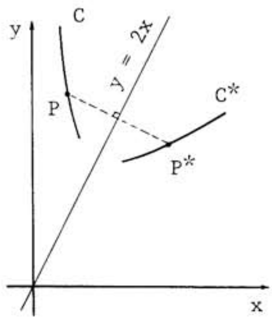
and for $3 \le k \le 15$,

$$x^k: \quad -c_{k-2} - c_{k-1} + c_k = 0.$$

It follows that for $k \le 15$, $c_k = F_{k+1}$, the $(k+1)$-st Fibonacci number. Thus $a = c_{15} = F_{16} = 987$.

14. (084)

Let $P(u,v)$ be any point on C, and let $P^*(x,y)$ be the corresponding point on C^*, that is, the reflection of P in the line $y = 2x$. Connect P and P^* by a straight line segment as shown in the figure. Then the problem is to find an equation relating x and y.



Since PP^* is perpendicular to the line $y = 2x$, its slope is $-\frac{1}{2}$. Thus

(1) $$\frac{y - v}{x - u} = -\frac{1}{2}.$$

Furthermore, since the midpoint of PP^* is on the line $y = 2x$, its coordinates must satisfy the equation of this line; that is,

(2) $$\frac{y + v}{2} = 2\cdot\frac{x + u}{2}.$$

Solving (1) and (2) simultaneously for u and v yields $u = (4y - 3x)/5$ and $v = (4x + 3y)/5$. Substituting these into $uv = 1$ (which holds since P is on C) we find that

$$12x^2 - 7xy - 12y^2 + 25 = 0.$$

This is the equation of C^* in the form desired; in it $bc = (-7)(-12) = 84$.

15. (704)

At any given time, the letters in the box are in decreasing order from top to bottom. Thus the sequence of letters in the box is uniquely determined by the set of letters in the box. We have two cases: letter 9 arrived before lunch or it did not.

Case 1. Since letter 9 arrived before lunch, no further letters will arrive, and the number of possible orders is simply the number of subsets of $T = \{1, 2, \ldots, 6, 7, 9\}$ which might still be in the box. In fact, each subset of T is possible, because the secretary might have typed letters not in the subset as soon as they arrived and not typed any others. Since T has 8 elements, it has $2^8 = 256$ subsets (including the empty set).

Case 2. Since letter 9 didn't arrive before lunch, the question is: where can it be inserted in the typing order? Any position is possible for each subset of $U = \{1, 2, \ldots, 6, 7\}$ which might have been left in the box during lunch (in descending order). For instance, if the letters in the box during lunch are 6,3,2, then the typing order 6,3,9,2 would occur if the boss would deliver letter 9 just after letter 3 was typed. There would seem to be $k+1$ places at which letter 9 could be inserted into a sequence of k letters. However, if letter 9 is inserted at the beginning of the sequence (i.e., at the top of the pile, so it arrives before any after-lunch typing is done), then we are duplicating an ordering from Case 1. Thus if k letters are in the basket after returning from lunch, then there are k places to insert letter 9 (without duplicating any Case 1 orderings). Thus we obtain

$$\sum_{k=0}^{7} k\binom{7}{k} = 7(2^{7-1}) = 448$$

new orderings in Case 2.

Combining these cases gives $256 + 448 = 704$ possible typing orders.

Note. The reasoning in Case 2 can be extended to cover both cases by observing that in any sequence of k letters not including letter 9, there are $k+2$ places to insert letter 9, counting the possibility of not having to insert it (i.e., if it arrived before lunch) as one of the cases. This yields

$$\sum_{k=0}^{7} (k+2)\binom{7}{k} = 704$$

possible orderings, in agreement with the answer found previously.