AMERICAN MATHEMATICS COMPETITIONS

9th ANNUAL AMERICAN INVITATIONAL MATHEMATICS EXAMINATION (AIME)

TUESDAY, MARCH 19, 1991

Sponsored by

Mathematical Association of America
Society of Actuaries   Mu Alpha Theta
National Council of Teachers of Mathematics
Casualty Actuarial Society   American Statistical Association
American Mathematical Association of Two-Year Colleges
American Mathematical Society

1. DO NOT OPEN THIS BOOKLET UNTIL TOLD TO DO SO BY YOUR PROCTOR.
2. **This is a 15-question, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers; i.e., there is neither partial credit nor a penalty for wrong answers.**
3. **Scratch paper, graph paper, ruler, compass, protractor and eraser are permitted.** *Calculators and slide rules are not permitted.*
4. **Please print the following:**

Name:____
Last   First   Middle initial

Home address:____
Street Address

____
City   State or Province   Zip or Postcode

_____________________________   ______   ______
Home Phone including Area Code   Sex (M or F)   Your age

_____________________________   ______
Full Name of School   Grade Level (e.g.,11)

5. Citizenship Status: USA Citizen_____ Permanent Resident_____ Other_____
   If other, explain: ________________________________
6. My score on the 1991 AHSME was ☐   I took the 1991 AHSME on (date):______________
7. A combination of the AIME and AHSME scores is used to determine eligibility for participation in the U. S. A. Mathematical Olympiad (USAMO) to be given on April 23, 1991. Please check one box:

   If I qualify for the USAMO, I agree to take it.   YES ☐   NO ☐

   (*Your school must also agree to administer the USAMO before you can take it.*)
8. Record all your answers, and certain other information, on the AIME answer form. Your Examination Manager will instruct you how to complete the form. Only the answer form and this cover will be collected from you.

1. Find $x^2 + y^2$ if x and y are positive integers such that

$$xy + x + y = 71 \qquad \text{and} \qquad x^2y + xy^2 = 880.$$

2. Rectangle $ABCD$ has sides $\overline{AB}$ of length 4 and $\overline{CB}$ of length 3. Divide $\overline{AB}$ into 168 congruent segments with points $A = P_0, P_1, \ldots, P_{168} = B$, and divide $\overline{CB}$ into 168 congruent segments with points $C = Q_0, Q_1, \ldots, Q_{168} = B$. For $1 \le k \le 167$, draw the segments $\overline{P_kQ_k}$. Repeat this construction on the sides $\overline{AD}$ and $\overline{CD}$, and then draw the diagonal $\overline{AC}$. Find the sum of the lengths of the 335 parallel segments drawn.

3. Expanding $(1 + 0.2)^{1000}$ by the binomial theorem and doing no further manipulation gives

$$\binom{1000}{0}(0.2)^0 + \binom{1000}{1}(0.2)^1 + \binom{1000}{2}(0.2)^2 + \cdots + \binom{1000}{1000}(0.2)^{1000}$$
$$= A_0 + A_1 + A_2 + \cdots + A_{1000},$$

where $A_k = \binom{1000}{k}(0.2)^k$ for $k = 0, 1, 2, \ldots, 1000$. For which k is A_k the largest?

4. How many real numbers x satisfy the equation $\frac{1}{5}\log_2 x = \sin(5\pi x)$?

5. Given a rational number, write it as a fraction in lowest terms and calculate the product of the resulting numerator and denominator. For how many rational numbers between 0 and 1 will 20! be the resulting product?

6. Suppose r is a real number for which

$$\left\lfloor r + \frac{19}{100} \right\rfloor + \left\lfloor r + \frac{20}{100} \right\rfloor + \left\lfloor r + \frac{21}{100} \right\rfloor + \cdots + \left\lfloor r + \frac{91}{100} \right\rfloor = 546.$$

Find $\lfloor 100r \rfloor$. (For real x, $\lfloor x \rfloor$ is the greatest integer less than or equal to x.)

7. Find A^2, where A is the sum of the absolute values of all roots of the following equation:

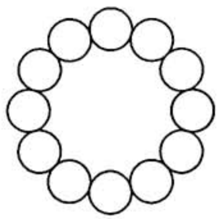
$$x = \sqrt{19}+\cfrac{91}{\sqrt{19}+\cfrac{91}{\sqrt{19}+\cfrac{91}{\sqrt{19}+\cfrac{91}{\sqrt{19}+\cfrac{91}{x}}}}}$$

8. For how many real numbers a does the quadratic equation $x^2 + ax + 6a = 0$ have only integer roots for x?

9. Suppose that $\sec x + \tan x = \frac{22}{7}$ and that $\csc x + \cot x = \frac{m}{n}$, where $\frac{m}{n}$ is in lowest terms. Find $m + n$.

10. Two three-letter strings, aaa and bbb, are transmitted electronically. Each string is sent letter by letter. Due to faulty equipment, each of the six letters has a 1/3 chance of being received incorrectly, as an a when it should have been a b, or as a b when it should have been an a. However, whether a given letter is received correctly or incorrectly is independent of the reception of any other letter. Let S_a be the three-letter string received when aaa is transmitted and let S_b be the three-letter string received when bbb is transmitted. Let p be the probability that S_a comes before S_b in alphabetical order. When p is written as a fraction in lowest terms, what is its numerator?

11. Twelve congruent disks are placed on a circle C of radius 1 in such a way that the twelve disks cover C, no two of the disks overlap, and so that each of the twelve disks is tangent to its two neighbors. The resulting arrangement of disks is shown in the figure to the right. The sum of the areas of the twelve disks can be written in the form $\pi(a - b\sqrt{c})$, where a, b, c are positive integers and c is not divisible by the square of any prime. Find $a + b + c$.



12. Rhombus $PQRS$ is inscribed in rectangle $ABCD$ so that vertices P, Q, R, and S are interior points on sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$, respectively. It is given that $PB = 15$, $BQ = 20$, $PR = 30$, and $QS = 40$. Let m/n, in lowest terms, denote the perimeter of $ABCD$. Find $m+n$.

13. A drawer contains a mixture of red socks and blue socks, at most 1991 in all. It so happens that, when two socks are selected randomly without replacement, there is a probability of exactly 1/2 that both are red or both are blue. What is the largest possible number of red socks in the drawer that is consistent with this data?

14. A hexagon is inscribed in a circle. Five of the sides have length 81 and the sixth, denoted by $\overline{AB}$, has length 31. Find the sum of the lengths of the three diagonals that can be drawn from A.

15. For positive integer n, define S_n to be the minimum value of the sum

$$\sum_{k=1}^{n} \sqrt{(2k-1)^2 + a_k^2},$$

where $a_1, a_2, \ldots, a_n$ are positive real numbers whose sum is 17. There is a unique positive integer n for which S_n is also an integer. Find this n.

SOLUTIONS

A 1991 Solutions Pamphlet will be sent to exam managers within a few weeks.

WRITE TO US!

Questions and comments about the problems and solutions for this AIME (but **not** requests for the Solutions Pamphlet) should be addressed to:

Prof Elgin H Johnston, AIME Chairman
Department of Mathematics
Iowa State University, Ames, IA 50011 USA

Comments about administrative arrangements and orders for any publications listed below should be addressed to:

Prof Walter E Mientka, AMC Executive Director
Department of Mathematics and Statistics
University of Nebraska, Lincoln, NE 68588-0322 USA

1991 USAMO

The USA Mathematical Olympiad is a 5-question, $3\frac{1}{2}$ hour, essay-type examination. The USAMO will be held on April 23, 1991. Your teacher has more details on who qualifies for the USAMO in the AHSME or AIME Teachers' Manuals. The best way to prepare for the USAMO is to study the exams from previous years and to review the contents of the ARBELOS. Copies may be ordered as indicated below.

PUBLICATIONS

MINIMUM ORDER: $5 (before handling fee), US FUNDS ONLY. Canada and US orders must be prepaid. Orders are mailed 4th class, unless you specify 1st class, in which case add $3.00 or 20% of total order, whichever is larger, with a maximum of $15.00. Make checks payable to the American Mathematics Competitions.

FOREIGN ORDERS: do **NOT** prepay; an invoice will be sent.

Examinations: Each price is for **one** copy of an exam **and** its solutions for one year. Specify the years you want and how many copies of each. All prices effective to September 1, 1991.

- **AJHSME** (Junior High Exam), 1985-1990, $1 per copy per year.
- **AHSME** 1972-91, $1 per copy per year.
- **AIME** 1983-91, $2 per copy per year.
- USA and International Mathematical **Olympiads** (together), 1976-90, $3 per copy per year.
- National **Summary** of Results and Awards, 1980-90, $4 per copy per year.

Books (Exams and solutions):

- Contest Problem Book I, AHSMEs 1950-60, $8.50.
- Contest Problem Book II, AHSMEs 1961-65, $8.50.
- Contest Problem Book III, AHSMEs 1966-72, $10.00.
- Contest Problem Book IV, AHSMEs 1973-82, $11.00.
- USA Mathematical Olympiads, 1972-86, $13.00.
- International Mathematical Olympiads, 1959-77, $10.00.
- International Mathematical Olympiads, 1978-85, $11.00.

Journal

- The ARBELOS (short articles and challenging problems); recommended especially for AIME and USAMO qualifiers. Five volumes plus a Geometry volume, $7.00 each.