AMERICAN MATHEMATICS COMPETITIONS

18^{th} ANNUAL (*ALTERNATE*) AMERICAN INVITATIONAL MATHEMATICS EXAMINATION

(AIME)

Tuesday, April 11, 2000

Sponsored by

Mathematical Association of America
Society of Actuaries   Mu Alpha Theta
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Clay Mathematics Institute   University of Nebraska-Lincoln
Kappa Mu Epsilon

1. DO NOT OPEN THIS BOOKLET UNTIL TOLD TO DO SO BY YOUR PROCTOR.

2. This is a 15-question, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers; i.e., there is neither partial credit nor a penalty for wrong answers.

3. No aids other than scratch paper, graph paper, ruler, compass, and protractor are permitted. In particular, **calculators are not permitted.**

4. A combination of the AIME and the American Mathematics Contest →10 or the American Mathematics Contest →12 scores are used to determine eligibility for participation in the U.S.A. Mathematical Olympiad (USAMO). The USAMO will be given on TUESDAY, May 2, 2000.

5. Record all of your answers, and certain other information, on the AIME answer form. Only the answer form will be collected from you.

1. The number

$$\frac{2}{\log_4 2000^6} + \frac{3}{\log_5 2000^6}$$

can be written as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m+n$.

2. A point whose coordinates are both integers is called a *lattice point.* How many lattice points lie on the hyperbola $x^2 - y^2 = 2000^2$?

3. A deck of forty cards consists of four 1's, four 2's, ..., and four 10's. A matching pair (two cards with the same number) is removed from the deck. Given that these cards are not returned to the deck, let m/n be the probability that two randomly selected cards also form a pair, where m and n are relatively prime positive integers. Find $m + n$.

4. What is the smallest positive integer with six positive odd integer divisors and twelve positive even integer divisors?

5. Given eight distinguishable rings, let n be the number of possible five-ring arrangements on the four fingers (not the thumb) of one hand. The order of rings on each finger is significant, but it is not required that each finger have a ring. Find the leftmost three nonzero digits of n.

6. One base of a trapezoid is 100 units longer than the other base. The segment that joins the midpoints of the legs divides the trapezoid into two regions whose areas are in the ratio 2:3. Let x be the length of the segment joining the legs of the trapezoid that is parallel to the bases and that divides the trapezoid into two regions of equal area. Find the greatest integer that does not exceed $x^2/100$.

7. Given that
$$\frac{1}{2!17!}+\frac{1}{3!16!}+\frac{1}{4!15!}+\frac{1}{5!14!}+\frac{1}{6!13!}+\frac{1}{7!12!}+\frac{1}{8!11!}+\frac{1}{9!10!}=\frac{N}{1!18!}$$
find the greatest integer that is less than $\frac{N}{100}$.

8. In trapezoid $ABCD$, leg $\overline{BC}$ is perpendicular to bases $\overline{AB}$ and $\overline{CD}$, and diagonals $\overline{AC}$ and $\overline{BD}$ are perpendicular. Given that $AB=\sqrt{11}$ and $AD=\sqrt{1001}$, find BC^2.

9. Given that z is a complex number such that $z+\frac{1}{z}=2\cos 3°$, find the least integer that is greater than $z^{2000}+\frac{1}{z^{2000}}$.

10. A circle is inscribed in quadrilateral $ABCD$, tangent to $\overline{AB}$ at P and to $\overline{CD}$ at Q. Given that $AP=19$, $PB=26$, $CQ=37$, and $QD=23$, find the square of the radius of the circle.

11. The coordinates of the vertices of isosceles trapezoid $ABCD$ are all integers, with $A=(20,100)$ and $D=(21,107)$. The trapezoid has no horizontal or vertical sides, and $\overline{AB}$ and $\overline{CD}$ are the only parallel sides. The sum of the absolute values of all possible slopes for $\overline{AB}$ is m/n, where m and n are relatively prime positive integers. Find $m+n$.

12. The points A, B and C lie on the surface of a sphere with center O and radius 20. It is given that $AB=13$, $BC=14$, $CA=15$, and that the distance from O to triangle ABC is $\frac{m\sqrt{n}}{k}$, where m, n, and k are positive integers, m and k are relatively prime, and n is not divisible by the square of any prime. Find $m+n+k$.

13. The equation $2000x^6+100x^5+10x^3+x-2=0$ has exactly two real roots, one of which is $\dfrac{m+\sqrt{n}}{r}$, where m, n and r are integers, m and r are relatively prime, and $r>0$. Find $m+n+r$.

14. Every positive integer k has a unique *factorial base expansion* $(f_1, f_2, f_3, \ldots, f_m)$, meaning that

$$k = 1! \cdot f_1 + 2! \cdot f_2 + 3! \cdot f_3 + \cdots + m! \cdot f_m,$$

where each f_i is an integer, $0 \le f_i \le i$, and $0 < f_m$. Given that $(f_1, f_2, f_3, \ldots, f_j)$ is the factorial base expansion of

$$16! - 32! + 48! - 64! + \cdots + 1968! - 1984! + 2000!,$$

find the value of $f_1 - f_2 + f_3 - f_4 + \cdots + (-1)^{j+1} f_j$.

15. Find the least positive integer n such that

$$\frac{1}{\sin 45^\circ \sin 46^\circ} + \frac{1}{\sin 47^\circ \sin 48^\circ} + \ldots + \frac{1}{\sin 133^\circ \sin 134^\circ} = \frac{1}{\sin n^\circ}$$