THE MATHEMATICAL ASSOCIATION OF AMERICA
AMERICAN MATHEMATICS COMPETITIONS

27th Annual *(Alternate)*

AMERICAN INVITATIONAL MATHEMATICS EXAMINATION

(AIME II)

Wednesday, April 1, 2009

1. DO NOT OPEN THIS BOOKLET UNTIL YOUR PROCTOR GIVES THE SIGNAL TO BEGIN.

2. This is a 15-question, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers; i.e., there is neither partial credit nor a penalty for wrong answers.

3. No aids other than scratch paper, graph paper, ruler, compass, and protractor are permitted. In particular, **calculators and computers are not permitted.**

4. A combination of the AIME and the American Mathematics Contest 10 or the American Mathematics Contest 12 scores are used to determine eligibility for participation in the U.S.A. Mathematical Olympiad (USAMO). The USAMO will be given on TUESDAY and WEDNESDAY, April 28-29, 2009.

5. Record all of your answers, and certain other information, on the AIME answer form. Only the answer form will be collected from you.

1. Before starting to paint, Bill had 130 ounces of blue paint, 164 ounces of red paint, and 188 ounces of white paint. Bill painted four equally sized stripes on a wall, making a blue stripe, a red stripe, a white stripe, and a pink stripe. Pink is a mixture of red and white, not necessarily in equal amounts. When Bill finished, he had equal amounts of blue, red, and white paint left. Find the total number of ounces of paint Bill had left.

2. Suppose that a, b, and c are positive real numbers such that $a^{\log_3 7} = 27$, $b^{\log_7 11} = 49$, and $c^{\log_{11} 25} = \sqrt{11}$. Find
$$a^{(\log_3 7)^2} + b^{(\log_7 11)^2} + c^{(\log_{11} 25)^2}.$$

3. In rectangle $ABCD$, $AB = 100$. Let E be the midpoint of $\overline{AD}$. Given that line AC and line BE are perpendicular, find the greatest integer less than AD.

4. A group of children held a grape-eating contest. When the contest was over, the winner had eaten n grapes, and the child in kth place had eaten $n + 2 - 2k$ grapes. The total number of grapes eaten in the contest was 2009. Find the smallest possible value of n.

5. Equilateral triangle T is inscribed in circle A, which has radius 10. Circle B with radius 3 is internally tangent to circle A at one vertex of T. Circles C and D, both with radius 2, are internally tangent to circle A at the other two vertices of T. Circles B, C, and D are all externally tangent to circle E, which has radius $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.
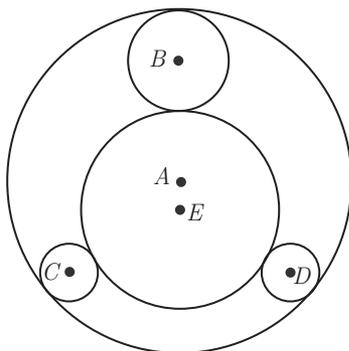
6. Let m be the number of five-element subsets that can be chosen from the set of the first 14 natural numbers so that at least two of the five numbers are consecutive. Find the remainder when m is divided by 1000.

7. Define $n!!$ to be $n(n-2)(n-4)\ldots 3\cdot 1$ for n odd and $n(n-2)(n-4)\ldots 4\cdot 2$ for n even. When $\displaystyle\sum_{i=1}^{2009}\frac{(2i-1)!!}{(2i)!!}$ is expressed as a fraction in lowest terms, its denominator is $2^a b$ with b odd. Find $\dfrac{ab}{10}$.

8. Dave rolls a fair six-sided die until a six appears for the first time. Independently, Linda rolls a fair six-sided die until a six appears for the first time. Let m and n be relatively prime positive integers such that $\frac{m}{n}$ is the probability that the number of times Dave rolls his die is equal to or within one of the number of times Linda rolls her die. Find $m+n$.

9. Let m be the number of solutions in positive integers to the equation $4x+3y+2z=2009$, and let n be the number of solutions in positive integers to the equation $4x+3y+2z=2000$. Find the remainder when $m-n$ is divided by 1000.

10. Four lighthouses are located at points A, B, C, and D. The lighthouse at A is 5 kilometers from the lighthouse at B, the lighthouse at B is 12 kilometers from the lighthouse at C, and the lighthouse at A is 13 kilometers from the lighthouse at C. To an observer at A, the angle determined by the lights at B and D and the angle determined by the lights at C and D are equal. To an observer at C, the angle determined by the lights at A and B and the angle determined by the lights at D and B are equal. The number of kilometers from A to D is given by $\dfrac{p\sqrt{r}}{q}$, where p, q, and r are relatively prime positive integers, and r is not divisible by the square of any prime. Find $p+q+r$.

11. For certain pairs (m,n) of positive integers with $m\geq n$ there are exactly 50 distinct positive integers k such that $|\log m-\log k|<\log n$. Find the sum of all possible values of the product mn.

12. From the set of integers $\{1,2,3,\ldots,2009\}$, choose k pairs $\{a_i,b_i\}$ with $a_i<b_i$ so that no two pairs have a common element. Suppose that all the sums a_i+b_i are distinct and less than or equal to 2009. Find the maximum possible value of k.

13. Let A and B be the endpoints of a semicircular arc of radius 2. The arc is divided into seven congruent arcs by six equally spaced points C_1, $C_2,\ldots,$ C_6. All chords of the form $\overline{AC_i}$ or $\overline{BC_i}$ are drawn. Let n be the product of the lengths of these twelve chords. Find the remainder when n is divided by 1000.

14. The sequence (a_n) satisfies $a_0 = 0$ and $a_{n+1} = \dfrac{8}{5}a_n + \dfrac{6}{5}\sqrt{4^n - a_n^2}$ for $n \geq 0$. Find the greatest integer less than or equal to a_{10}.

15. Let $\overline{MN}$ be a diameter of a circle with diameter 1. Let A and B be points on one of the semicircular arcs determined by $\overline{MN}$ such that A is the midpoint of the semicircle and $MB = \frac{3}{5}$. Point C lies on the other semicircular arc. Let d be the length of the line segment whose endpoints are the intersections of diameter $\overline{MN}$ with the chords $\overline{AC}$ and $\overline{BC}$. The largest possible value of d can be written in the form $r - s\sqrt{t}$, where r, s, and t are positive integers and t is not divisible by the square of any prime. Find $r + s + t$.

Your Exam Manager will receive a copy of the 2009 AIME Solution Pamphlet with the scores.

CONTACT US -- *Correspondence about the problems and solutions for this AIME and orders for any of our publications should be addressed to:*

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2009 USAMO -- THE USA MATHEMATICAL OLYMPIAD (USAMO) is a 6-question, 9-hour, essay-type examination. The USAMO will be held in your school on Tuesday and Wednesday, April 28-29, 2009. Your teacher has more details on who qualifies for the USAMO in the AMC 10/12 and AIME Teachers' Manuals. The best way to prepare for the USAMO is to study previous years of these exams, the World Olympiad Problems/Solutions and review the contents of the Arbelos. Copies may be ordered from the web sites indicated below.

PUBLICATIONS -- For a complete listing of available publications please visit the following web sites:

AMC - - http://www.unl.edu/amc/d-publication/publication.shtml
MAA -- https://enterprise.maa.org/ecomtpro/timssnet/common/tnt_frontpage.cfm

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