AMERICAN MATHEMATICS COMPETITIONS

AIME SOLUTIONS PAMPHLET
FOR STUDENTS AND TEACHERS

10th ANNUAL
AMERICAN INVITATIONAL MATHEMATICS EXAMINATION
(AIME)

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This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs. geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers will share these solutions with their students, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution. *Remember that reproduction of these solutions is prohibited by copyright.*

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1. (Answer: 400)
A positive rational number that is less than 10 and has denominator 30 can be written in the form
$$\frac{30n+r}{30},$$
where n and r are integers satisfying $0 \le n \le 9$ and $0 \le r < 30$. Furthermore, such a fraction is in lowest terms if and only if r and 30 are relatively prime; i.e., if and only if $r \in \{1,7,11,13,17,19,23,29\}$. Thus there are 10 choices for n and 8 choices for r, and no two pairs of choices (n,r) give the same value of $(30n+r)/30$. It follows that the desired sum has $8 \cdot 10 = 80$ terms. These may be paired by noting that $k/30$ is one of these fractions if and only if $10 - k/30 = (300-k)/30$ is as well. Since the sum of each of these pairs is 10, we find that the sum of all such fractions is
$$\left(\frac{1}{30}+\frac{300-1}{30}\right)+\left(\frac{7}{30}+\frac{300-7}{30}\right)+\cdots+\left(\frac{149}{30}+\frac{300-149}{30}\right)=40\cdot 10=400.$$

2. (Answer: 502)
An ascending positive integer must have distinct, nonzero digits. Thus the digits must be a subset of two or more elements from the set $S = \{1,2,3,4,5,6,7,8,9\}$. Conversely, any subset of S that has two or more elements corresponds to a unique ascending positive integer in which the elements of the subset are arranged in increasing order. It follows that the number of ascending positive integers is equal to the number of subsets of S that have two or more elements. Since a nine-element set has $2^9 = 512$ subsets and ten of these subsets have fewer than two elements, the number of ascending positive integers is $512 - 10 = 502$.

3. (Answer: 164)
Let

$W =$ the player's number of wins at the start of the weekend

and

$M =$ the number of matches played at the start of the weekend.

We are given $W/M = .500$ and $(W+3)/(M+4) > .503$. Thus $M = 2W$ and
$$W + 3 > .503(2W+4) = 1.006W + 2.012.$$
It follows that $W < (3-2.012)/.006 = 164.\overline{6}$. Finally, note that if $W = 164$ and $M = 328$, then $W/M = .500$ and $(W+3)/(M+4) > .503$. Hence, the largest number of matches that the player could have won before the start of the weekend is 164.

4. (Answer: 062)
Row n of Pascal's triangle consists of the binomial coefficients $\binom{n}{k}$, $k = 0, 1, \ldots, n$. If three consecutive entries in row n of Pascal's triangle are in the ratio $3:4:5$, then there is a positive integer k for which

$$\frac{3}{4} = \frac{\binom{n}{k-1}}{\binom{n}{k}} = \frac{\frac{n!}{(k-1)!(n-k+1)!}}{\frac{n!}{k!(n-k)!}} = \frac{k!(n-k)!}{(k-1)!(n-k+1)!} = \frac{k}{n-k+1}$$

and

$$\frac{4}{5} = \frac{\binom{n}{k}}{\binom{n}{k+1}} = \frac{\frac{n!}{k!(n-k)!}}{\frac{n!}{(k+1)!(n-k-1)!}} = \frac{(k+1)!(n-k-1)!}{k!(n-k)!} = \frac{k+1}{n-k}.$$

It follows that

$$3n - 7k = -3 \qquad \text{and} \qquad 4n - 9k = 5.$$

Solving simultaneously gives $k = 27$ and $n = 62$. Thus, the consecutive entries $\binom{62}{26}$, $\binom{62}{27}$, $\binom{62}{28}$ in row 62 of Pascal's triangle are in the ratio $3:4:5$.

5. (Answer: 660)
First note that

$$0.\overline{abc} = \frac{abc}{999},$$

and that $999 = 3^3 37$.
If abc is divisible by neither 3 nor 37, then this fraction is already in lowest terms. By the Inclusion-Exclusion Principle, there are

$$999 - \left(\frac{999}{3} + \frac{999}{37}\right) + \left(\frac{999}{3 \cdot 37}\right) = 999\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{37}\right) = 648$$

such numbers.
Some of the reduced fractions may have numerators that are divisible by 3 or 37. Such fractions must have the form

$$\frac{k}{37}\text{, where } k \text{ is a multiple of 3 but not a multiple of 37}$$

or

$$\frac{l}{3^m}\text{, where } l \text{ is a multiple of 37 but not a multiple of 3, and } m = 1,\ 2,\ 3.$$

There are no fractions of the second type in S, since any fraction of this form is greater than 1. There are 12 fractions of the first type in S, one for each of $k = 3, 6, \ldots, 36$. Thus the number of distinct numerators in the set of reduced fractions is $648 + 12 = 660$.

6. (Answer: 156)

Let n have decimal representation $1abc$. If one of a, b, or c is 5, 6, 7, or 8, then there will be carrying when n and $n+1$ are added. If $b=9$ and $c \neq 9$, or if $a=9$ and either $b \neq 9$ or $c \neq 9$, there will also be carrying when n and $n+1$ are added.

If n is not one of the integers described above, then n has one of the forms

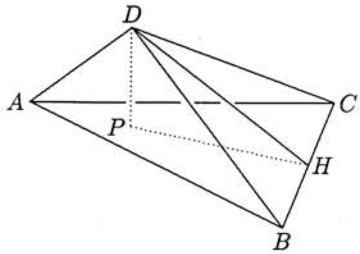
$$1abc \qquad 1ab9 \qquad 1a99 \qquad 1999,$$

where $a, b, c \in \{0, 1, 2, 3, 4\}$. For such n, no carrying will be needed when n and $n+1$ are added. There are $5^3+5^2+5+1=156$ such values of n.

7. (Answer: 320)

Let H be the foot of the perpendicular from D to line BC. Since $BC=10$ and the area of $\triangle BCD$ is 80, $DH=16$. Next, let P be the foot of the perpendicular from D to plane ABC. Then $\triangle HPD$ is a right triangle and $\angle DHP$ measures 30°. It follows that $DP=\frac{1}{2}DH=8$, and the volume of the tetrahedron is†

$$\frac{1}{3}DP \cdot [ABC] = \frac{1}{3} \cdot 8 \cdot 120 = 320.$$



8. (Answer: 819)

Suppose that the first term of the sequence ΔA is d. Then the sequence ΔA is $(d,\, d+1,\, d+2, \ldots)$ with n^{th} term given by $d+(n-1)$. Hence the sequence A is

$$\left(a_1,\ a_1+d,\ a_1+d+(d+1),\ a_1+d+(d+1)+(d+2),\ \ldots\ \right),$$

with n^{th} term given by

$$a_n = a_1 + (n-1)d + \frac{1}{2}(n-1)(n-2).$$

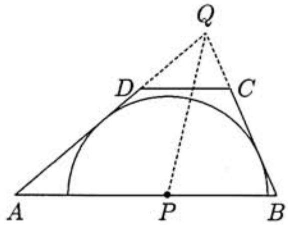
This shows that a_n is a quadratic polynomial in n with leading coefficient $\frac{1}{2}$. Since $a_{19}=a_{92}=0$, we must have

$$a_n = \frac{1}{2}(n-19)(n-92),$$

so $a_1 = \frac{1}{2}(1-19)(1-92) = 819$.

† The area of triangle ABC is denoted $[ABC]$.

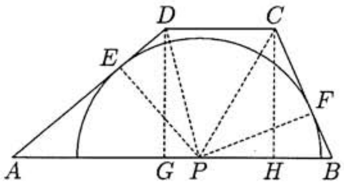
9. (Answer: 164)
Extend $\overline{AD}$ and $\overline{BC}$ until the two segments meet at a point Q. Point P is equidistant from $\overline{AQ}$ and $\overline{BQ}$, so P lies on the bisector of $\angle AQB$. Since the bisector of the angle of a triangle divides the side opposite the angle into segments whose lengths are proportional to the lengths of the adjoining sides, we have



$$\frac{AP}{BP} = \frac{AQ}{BQ} = \frac{AD}{BC} = \frac{7}{5}.$$

Since $AP + PB = 92$, it follows that $AP = 161/3$ and $m + n = 164$.

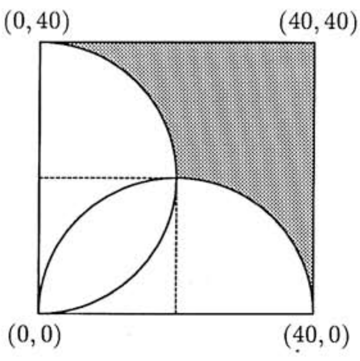
Alternate Solution. Label as E the point at which the circle is tangent to $\overline{AD}$, and as F the point at which the circle is tangent to $\overline{BC}$. Let G be the foot of the perpendicular from D to $\overline{AB}$, and H the foot of the perpendicular from C to $\overline{AB}$. Since $PE = PF$ and $DG = CH$, it follows that



$$\frac{AP}{BP} = \frac{\frac{1}{2}AP \cdot DG}{\frac{1}{2}BP \cdot CH} = \frac{[APD]}{[PBC]}$$
$$= \frac{\frac{1}{2}AD \cdot EP}{\frac{1}{2}BC \cdot FP} = \frac{AD}{BC} = \frac{7}{5}.$$

Since $AP + PB = 92$, it follows that $AP = 161/3$ and $m + n = 164$.

10. (Answer: 572)
Let $z = x + iy$. For z to be in the region in question, we must have $0 \le x \le 40$ and $0 \le y \le 40$. Hence the region lies in the square with vertices $(0,0)$, $(40,0)$, $(40,40)$, and $(0,40)$. Next note that



$$\frac{40}{\overline{z}} = \frac{40}{x - iy} = \frac{40x}{x^2+y^2} + i\frac{40y}{x^2+y^2}.$$

Hence the restrictions on the real and imaginary parts of $40/\overline{z}$ give

$$0 \le \frac{40x}{x^2+y^2} \le 1 \quad \text{and} \quad 0 \le \frac{40y}{x^2+y^2} \le 1,$$

from which

$$(x-20)^2+y^2 \ge 20^2 \quad \text{and} \quad x^2+(y-20)^2 \ge 20^2.$$

Thus the region in question lies outside the circle with center $(20, 0)$ and radius 20 and also outside the circle with center $(0, 20)$ and radius 20, as indicated by the shaded portion of the diagram. As suggested by the dashed lines in the diagram, the area of the region is $3/4$ the area of the square minus the area of two quarter-circles. Hence

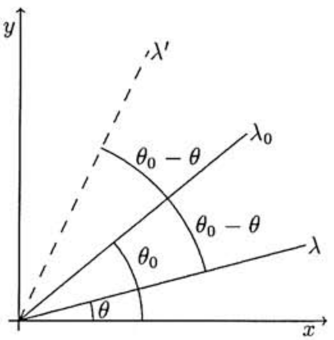
$$\text{Area}(A) = \frac{3}{4} \cdot 40^2 - \frac{2}{4}(\pi \cdot 20^2) = 200(6 - \pi) \approx 571.7,$$

so that the desired number is 572.

11. (Answer: 945)
Let λ_0 and λ be lines through the origin making angles of θ_0 and θ, respectively, with the positive x-axis. When λ is reflected in λ_0, the resulting line λ' makes an angle of

$$\theta_0 + (\theta_0 - \theta) = 2\theta_0 - \theta$$

with the positive x-axis. Thus, if λ is reflected in ℓ_1, then the result is a line λ_1 that passes through the origin and makes an angle of $2\frac{\pi}{70} - \theta$ with the positive x-axis. Reflecting λ_1 in the line ℓ_2 gives a line λ_2 through the origin that makes an angle of



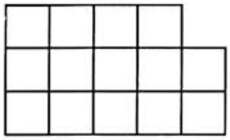
$$2\frac{\pi}{54} - \left(2\frac{\pi}{70} - \theta\right) = -\frac{8\pi}{945} + \theta$$

with the positive x-axis. Thus $R(\lambda)$ is obtained by rotating λ through $-8\pi/945$ radians and $R^{(m)}(\lambda)$ is obtained by rotating λ through $-8m\pi/945$ radians. For $R^{(m)}(\lambda) = \lambda$ to hold, $8m/945$ must be an integer. The smallest positive integer value of m for which this is true is 945.

Queries. If the line ℓ did not pass through the origin, would the above answer be affected? If R were defined by consecutive reflection about an odd number of lines through the origin, then m would be either 1 or 2. Why?

12. (Answer: 792)

At any stage of the game, the uneaten squares will form columns of non-increasing heights as we read from left to right. It is not hard to show that this condition is not only necessary, but is also sufficient for a given configuration of squares to occur in a game. (The reader should prove this fact.) Moreover, any such configuration can be completely described by the twelve-step polygonal path that runs from the upper left to the lower right of the original board, forming the boundary between the eaten and uneaten squares. This polygonal boundary can be described by a twelve-letter sequence of **V**'s and **H**'s . Such a sequence contains seven **H**'s, where each **H** represents the top of an uneaten column (or the bottom of a completely eaten one) and five **V**'s, where each **V** represents a one-unit drop in vertical height in moving from the top of an uneaten column to the top of an adjacent, but shorter column. For example, the state that appears in the diagram accompanying the problem is described by **HHHVHVVHHHVV**, while the state in the diagram above is given by **VVHHHHVHVVHH**. There are $\frac{12!}{7!5!} = 792$ sequences of seven **H**'s and five **V**'s, including the sequences **HHHHHHHVVVVV** and **VVVVVHHHHHHH**, which describe the full board and the empty board, respectively.



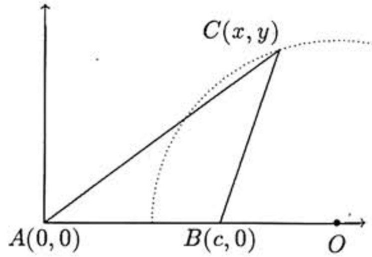
Note. The game of *Chomp* is due to David Gale, and was introduced (and named) by Martin Gardner in his *Scientific American* column "Mathematical Games". The column reappeared in Gardner's collection *Knotted Doughnuts*.

13. (Answer: 820)

Let $AB = c$, $AC = br$, and $BC = ar$, with $a < b$. We shall show that the locus of all such points C is a circle whose center is on line AB and whose radius is $abc/(b^2 - a^2)$. (This circle is called a circle of Apollonius.) The radius of the circle serves as the height of the triangle of maximal area, so the desired area is

$$\frac{1}{2}c\left(\frac{abc}{b^2-a^2}\right).$$

Taking $a : b = 40 : 41$ and $c = 9$, we get an answer of 820.



One way to proceed is with coordinates: Let $A = (0, 0)$, $B = (c, 0)$, and $C = (x, y)$. Then $BC/AC = a/b$ becomes

$$\frac{\sqrt{(x-c)^2+y^2}}{\sqrt{x^2+y^2}} = \frac{a}{b}.$$

Squaring both sides and rearranging terms leads to

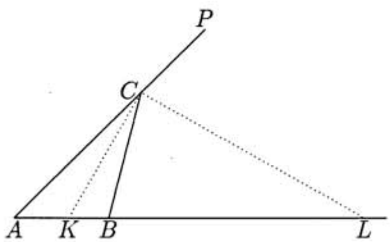
$$(b^2-a^2)x^2 - 2b^2cx + (b^2-a^2)y^2 = -b^2c^2.$$

Completing the square then gives

$$\left(x - \frac{b^2c}{b^2-a^2}\right)^2 + y^2 = \frac{a^2b^2c^2}{(b^2-a^2)^2}.$$

Hence the set of all vertices C satisfying the conditions of the problem is the circle of center $O = \left(\frac{b^2c}{b^2-a^2}, 0\right)$ and radius $\frac{abc}{b^2-a^2}$.

Alternate Solution. Assume $a < b$ and let K and L satisfy $AK : KB = b : a = AL : LB$, with K on $\overline{AB}$ and L on the extension of $\overline{AB}$ through B. Extend $\overline{AC}$ through C to P. Because $AC : CB = b : a$ also, the Angle Bisector Theorem implies that $\overline{CK}$ bisects angle ACB and $\overline{CL}$ bisects the exterior angle BCP. It follows that angle KCL is a right angle, so C lies on the circle that has KL as a diameter. It is straightforward to calculate that $KB = ac/(a + b)$ and that $BL = ac/(b - a)$. Therefore the radius of the circle is $abc/(b^2 - a^2)$, which serves as the altitude of the triangle ABC of maximal area. When $c = 9$ and $b : a = 41 : 40$, this area is 820.



Alternate Solution. Let $AB = c$, $AC = b$, and $BC = a$, with $a < b$. Then

$$[ABC] = \frac{1}{2}ab\sin C = \frac{1}{2}ab\sin C\left(\frac{\frac{c}{\sin C}\frac{c}{\sin C}}{\frac{a}{\sin A}\frac{b}{\sin B}}\right) = \frac{1}{2}c^2\frac{\sin A\sin B}{\sin C},$$

where we have used the Law of Sines for the second equality above. Since $C = \pi - A - B$, this equation can be rewritten as

$$[ABC] = \frac{1}{2}c^2\frac{\sin A\sin B}{\sin(A+B)} \le \frac{1}{2}c^2\frac{\sin A\sin B}{\sin(B+A)\sin(B-A)},$$

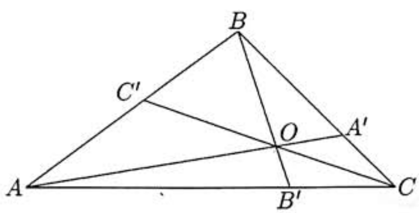
where equality holds if and only if $B = \frac{\pi}{2} + A$. Note that $B > A$ follows from $b > a$. Using the trigonometric identity $\sin(B+A)\sin(B-A) = \sin^2 B - \sin^2 A$ and then the Law of Sines again, we have

$$[ABC] \le \frac{1}{2}c^2\frac{\sin A\sin B}{\sin^2 B - \sin^2 A} = \frac{1}{2}c^2\frac{ab}{b^2 - a^2}.$$

If $c = 9$ and $b : a = 41 : 40$, we see that the maximum possible area is 820, and that this maximum is attained when $B - A = \frac{\pi}{2}$.

14. (Answer: 094)
Since $\triangle AOB$ and $\triangle A'OB$ share an altitude, as do $\triangle AOC$ and $\triangle A'OC$, we have

$$\frac{AO}{OA'} = \frac{[AOB]}{[A'OB]} = \frac{[COA]}{[COA']} = \frac{[AOB]+[COA]}{[A'OB]+[COA']} = \frac{[AOB]+[COA]}{[BOC]} = \frac{z+y}{x},$$



where $x = [BOC]$, $y = [COA]$, and $z = [AOB]$. Similarly,

$$\frac{BO}{OB'} = \frac{x+z}{y} \quad \text{and} \quad \frac{CO}{OC'} = \frac{y+x}{z}.$$

We then have

$$\begin{aligned}\frac{AO}{OA'}\frac{BO}{OB'}\frac{CO}{OC'} &= \frac{(z+y)(x+z)(y+x)}{xyz}\\ &= \frac{yz^2+y^2z+x^2z+xz^2+xy^2+x^2y+2xyz}{xyz}\\ &= \frac{yz(z+y)+xz(x+z)+xy(y+x)}{xyz}+2\\ &= \frac{z+y}{x}+\frac{x+z}{y}+\frac{y+x}{z}+2.\end{aligned}$$

Hence,

$$\frac{AO}{OA'}\frac{BO}{OB'}\frac{CO}{OC'} = \left(\frac{AO}{OA'}+\frac{BO}{OB'}+\frac{CO}{OC'}\right)+2 = 92+2 = 94.$$

15. (Answer: 396)
Let $f(m)$ be the number of ending zeros in the decimal expansion of $m!$. It is clear that $f(m)$ is a nondecreasing function of m. Furthermore, when m is a multiple of 5 we have

$$f(m) = f(m+1) = f(m+2) = f(m+3) = f(m+4) < f(m+5).$$

Thus if we list the numbers $f(k)$ for $k = 0, 1, 2, \ldots$, we obtain

$$0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, \ldots, 4, 4, 4, 4, 4, 6, 6, 6, 6, 6, \ldots, \qquad (*)$$

and each number in this list appears 5 times. We would like to know if the number 1991 appears in this list. It is well known (and easy to show) that the number of zeros at the end of $m!$ is

$$f(m) = \sum_{k=1}^{\infty}\left\lfloor \frac{m}{5^k}\right\rfloor.$$

If there is an m for which $f(m) = 1991$, then

$$1991 < \sum_{k=1}^{\infty}\frac{m}{5^k} = m\frac{\frac{1}{5}}{1-\frac{1}{5}} = \frac{m}{4}.$$

Hence $m > 4\cdot 1991 = 7964$. Using the above formula for $f(m)$ we find that $f(7965) = 1988$, and once this is known we can readily ascertain that $f(7975) = 1991$. Now if the list $(*)$ is carried out to the term $f(7979) = 1991$ we have

$$0, 0, 0, 0, 0, 1, 1, 1, 1, 1, \ldots, 1989, 1991, 1991, 1991, 1991, 1991.$$

This list contains 7980 terms, and each integer in the sequence occurs exactly 5 times. Thus the list has $7980/5 = 1596$ distinct integer values from the set $\{0, 1, 2, \ldots, 1991\}$. Hence $1992 - 1596 = 396$ of these integers do not appear in the list. Consequently there are 396 positive integers less than 1992 that are not factorial tails.