AMERICAN MATHEMATICS COMPETITIONS

AIME SOLUTIONS PAMPHLET
FOR STUDENTS AND TEACHERS

17th ANNUAL
AMERICAN INVITATIONAL MATHEMATICS EXAMINATION
(AIME)

TUESDAY, March 16, 1999

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This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs. geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution. *Remember that reproduction of these solutions is prohibited by copyright.*

This examination was prepared during the tenure of American Mathematics Competitions Executive Director, Dr. Walter E. Mientka.

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1. (Answer: 029)
The common difference of such a sequence must be even, for otherwise at least one of the first five terms would be even and greater than 2. The common difference must also be divisible by 3, for otherwise at least one of the first five terms would be divisible by 3 and greater than 3. Therefore the common difference must be divisible by 6, and the first term of the sequence must be relatively prime to 6. Because the sequence 5, 11, 17, 23, 29 consists exclusively of primes, it follows that the desired prime is 29.

2. (Answer: 118)
The line must go through the point where the diagonals of the parallelogram bisect each other, namely $\left(\dfrac{10+28}{2}, \dfrac{45+153}{2}\right) = (19, 99)$. Thus the slope of the line is $99/19$, and $m + n = 118$.

3. (Answer: 038)
If $n^2 - 19n + 99 = m^2$ for positive integers m and n, then $4m^2 = 4n^2 - 76n + 396 = (2n-19)^2 + 35$. Thus $4m^2 - (2n-19)^2 = 35$, or $(2m+2n-19)(2m-2n+19) = 35$. The sum of the two factors is $4m$, a positive integer, so the pair $(2m+2n-19, 2m-2n+19)$ can only be $(1, 35)$, $(5, 7)$, $(7, 5)$, or $(35, 1)$. Subtract the second factor from the first to discover that $4n - 38$ can be only -34, -2, 2, or 34, from which it follows that n can only be 1, 9, 10, or 18. The sum of these integers is 38.

4. (Answer: 185)
Notice that O is the center of the circle in which both squares are inscribed. The reflection of either square across the diameter determined by $\overline{OB}$ is another square inscribed in the same circle. Because the circle has only two chords of length 1 that go through B, the squares must be reflected images of each other. In particular, $AB = BC$. Similar reasoning shows that any two adjacent sides of $ABCDEFGH$ have the same length, so the octagon is equilateral. Because the distance from O to all eight sides of the squares is $1/2$, the area of the octagon is eight times the area of triangle AOB; i.e., $8(\frac{1}{2})(\frac{1}{2})AB = 2AB = 86/99$. Thus $m + n = 86 + 99 = 185$.

OR

Let J be that vertex of one of the squares for which angle AJB is right, and let $x = AJ$ and $y = BJ$. Then

$$x + y + \frac{43}{99} = 1, \qquad \text{so} \qquad x + y = \frac{56}{99}, \qquad \text{and} \qquad x^2 + y^2 = \left(\frac{43}{99}\right)^2.$$

Hence

$$2xy = (x+y)^2 - (x^2+y^2) = \frac{56^2 - 43^2}{99^2} = \frac{13}{99}.$$

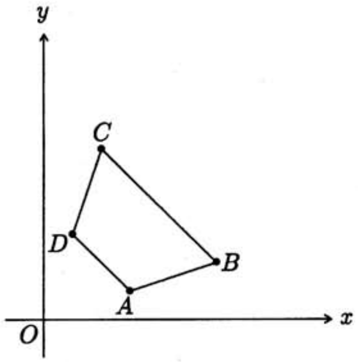
Because $2xy$ is the combined area of the four corner triangles, the area of the octagon is $1 - \frac{13}{99} = \frac{86}{99}$, and $m + n = 185$.

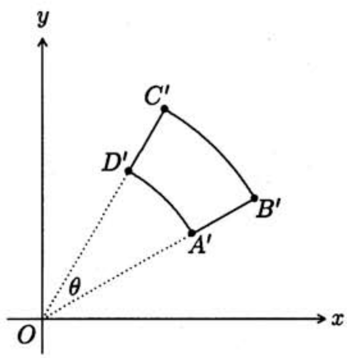
Query: Could this problem have been posed with the hypothesis $AB = 19/99$?

5. (Answer: 223)
If the final digit of x is less than 8, then $S(x+2) = 2 + S(x)$, so $T(x) = 2$. When the last digit of x is 8, then x has the form $A|B|8$, where B is a block of k nines, k is nonnegative, and the final digit of the block A is not 9. Because $x + 2$ has the form $(A+1)|Z$, where Z is a block of $k+1$ zeros, it follows that $T(x) = S(x) - S(x+2) = S(A) + 9k + 8 - S(A+1) = S(A) + 9k + 8 - S(A) - 1 = 9k + 7$. When the last digit of x is 9, then x has the form $A|B$, where B is a block of k nines, k is positive, and the final digit of A is not 9. Because $x+2$ has the form $(A+1)|Z|1$, where Z is a block of $k-1$ zeros, it follows that $T(x) = S(x) - S(x+2) = S(A) + 9k - S(A+1) - 1 = 9k - 2$. Notice that the sequence $9k + 7$ for nonnegative k coincides with the sequence $9k - 2$ for positive k. Thus T can have the values $7, 16, 25, \ldots, 1996$, and 2. There are $\frac{1}{9}(1996 - (-2)) + 1 = 223$ values in all.

6. (Answer: 314)

Notice that $\overline{AB}$ is contained in the line whose equation is $3y = x$. The image of a point on $\overline{AB}$ must therefore satisfy $3y^2 = x^2$. Because the coordinates of the image points must be positive, the image $\overline{A'B'}$ of $\overline{AB}$ is contained in the line $y\sqrt{3} = x$. In a similar fashion, it follows that the image $\overline{C'D'}$ of $\overline{CD}$ is contained in the line $y = x\sqrt{3}$. An equation for line AD is $x + y = 1200$, so the image of $\overline{AD}$ is contained in the first-quadrant part of the circle $x^2 + y^2 = 1200$. In a similar fashion, it follows that the image of $\overline{BC}$ is contained in the first-quadrant part of the circle $x^2 + y^2 = 2400$. Thus the area of the region enclosed by the image of quadrilateral $ABCD$ is $\frac{\theta}{360}\left(\pi(OB')^2 - \pi(OA')^2\right)$, where O is the origin and θ is the degree measure of the angle formed by $\overline{OA'}$ and $\overline{OD'}$. Notice that $\theta = \tan^{-1}\sqrt{3} - \tan^{-1}\frac{1}{\sqrt{3}} = 30$, because the slope of $\overline{OD'}$ is $\sqrt{3}$ and the slope of $\overline{OA'}$ is $\frac{1}{\sqrt{3}}$. Hence the area of the region enclosed by the image of $ABCD$ is $k = \frac{1}{12}(2400 - 1200)\pi = 100\pi$, and the greatest integer that does not exceed k is 314.



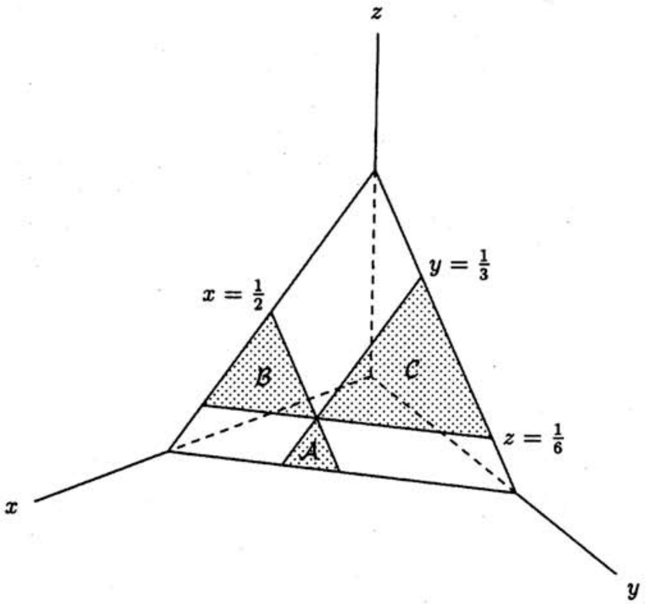


7. (Answer: 650)

A switch will finish in position A if and only if it has been advanced $4k$ times for some integer k. Each advance of a given switch corresponds to a multiple of its label. Let S be the set of integers $2^x3^y5^z$, where x, y, and z take on the values 0, 1, ..., 9. Notice that $2^x3^y5^z$ has $(10-x)(10-y)(10-z)$ multiples in S. Thus the answer to the problem is the number of triples (x,y,z) for which $(10-x)(10-y)(10-z)$ is divisible by 4. There are two cases in which $(10-x)(10-y)(10-z)$ is *not* divisible by 4. If all three factors of $(10-x)(10-y)(10-z)$ are odd, the product will also be odd; this occurs $5 \cdot 5 \cdot 5 = 125$ times. If two of the factors are odd and the third is 2, 6, or 10, the product will be even but not divisible by 4; this occurs $3 \cdot 5 \cdot 5 \cdot 3 = 225$ times. In all, there are $125 + 225 = 350$ triples (x,y,z) for which $(10-x)(10-y)(10-z)$ is not divisible by 4. Therefore after step 1000, the number of switches in position A will be $1000 - 350 = 650$.

8. (Answer: 025)

Notice that $\mathcal{T}$ is a first-octant equilateral triangle, whose vertices are $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. The planes $x=\frac{1}{2}$, $y=\frac{1}{3}$, and $z=\frac{1}{6}$ intersect $\mathcal{T}$ along line segments that are parallel to the sides of $\mathcal{T}$. Let $\mathcal{A}$ consist of those points of $\mathcal{T}$ that satisfy $x \geq \frac{1}{2}$ and $y \geq \frac{1}{3}$. Notice that $z \leq \frac{1}{6}$ for any point in $\mathcal{A}$, so the points of $\mathcal{A}$ support $\left(\frac{1}{2},\frac{1}{3},\frac{1}{6}\right)$, with the exception of $\left(\frac{1}{2},\frac{1}{3},\frac{1}{6}\right)$ itself. In a similar fashion, let $\mathcal{B}$ consist of those points of $\mathcal{T}$ that satisfy $x \geq \frac{1}{2}$ and $z \geq \frac{1}{6}$, and let $\mathcal{C}$ consist of those points of $\mathcal{T}$ that satisfy $y \geq \frac{1}{3}$ and $z \geq \frac{1}{6}$. Except for the point $\left(\frac{1}{2},\frac{1}{3},\frac{1}{6}\right)$, $\mathcal{S}$ is the union of the equilateral triangles $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, whose sides are $\frac{1}{6}$, $\frac{1}{3}$, and $\frac{1}{2}$ times as long as the sides of $\mathcal{T}$, and whose areas are $\frac{1}{36}$, $\frac{1}{9}$, and $\frac{1}{4}$ times the area of $\mathcal{T}$, respectively. It follows that the area of $\mathcal{S}$ divided by the area of $\mathcal{T}$ is $\frac{1}{36}+\frac{1}{9}+\frac{1}{4}=\frac{7}{18}$. Thus $m+n=25$.



9. (Answer: 259)

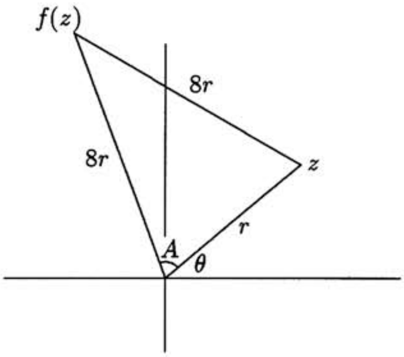
Because $(a+bi)z$ is equidistant from z and 0, $|(a+bi)z - z| = |(a+bi)z|$. Thus $|a-1+bi| = |a+bi|$, or $(a-1)^2 + b^2 = a^2 + b^2$, so $a = \dfrac{1}{2}$. Now use the information $|a+bi| = 8$ to deduce that $b^2 = 64 - \dfrac{1}{4} = \dfrac{255}{4}$, so that $m+n = 259$.

OR

Let $z = r\operatorname{cis}\theta = r(\cos\theta + i\sin\theta)$, where r is positive and $0 \le \theta < 2\pi$, and let $a + bi = 8\operatorname{cis} A$, where $0 < A < \pi/2$. Thus $f(z) = 8r\operatorname{cis}(\theta + A)$. Notice that $\cos A = \dfrac{r/2}{8r} = \dfrac{1}{16}$, because the triangle in the figure is isosceles. It follows that

$$b^2 = 8^2 \sin^2 A = 64(1 - \cos^2 A) = \frac{255}{4}.$$

Thus $m + n = 259$.



10. (Answer: 489)

Begin generally with k points in the plane, no three of which are collinear. There are $\binom{k}{2}$ segments joining the points, and $\binom{\binom{k}{2}}{4}$ ways to choose four segments. Because two triangles can share at most one side, four segments cannot form two triangles. Therefore, it suffices to count the ways of choosing a triangle and one additional segment. There are $\binom{k}{3}$ ways to choose the vertices of a triangle, and then $\binom{k}{2} - 3$ ways to choose an additional segment. Hence the probability of obtaining a triangle when $k = 10$ is

$$\frac{\binom{k}{3}\left(\binom{k}{2} - 3\right)}{\binom{\binom{k}{2}}{4}} = \frac{\binom{10}{3}\left(\binom{10}{2} - 3\right)}{\binom{45}{4}} = \frac{16}{473},$$

so that $m + n = 489$.

11. (Answer: 177)

$$\begin{aligned}\sum_{k=1}^{35} \sin 5k &= \frac{1}{\sin 5}\sum_{k=1}^{35} \sin 5 \sin 5k \\ &= \frac{1}{\sin 5}\sum_{k=1}^{35} \frac{\cos(5k-5)-\cos(5k+5)}{2} \\ &= \frac{1}{\sin 5}\cdot\frac{\cos 0+\cos 5-\cos 175-\cos 180}{2} \\ &= \frac{1-\cos 175}{\sin 175} = \tan\frac{175}{2},\end{aligned}$$

so $m+n=177$.

OR

Let $\operatorname{cis} t = \cos t + i \sin t$. Because $\operatorname{cis} 0 = 1$, the given series is the imaginary part of the complex series

$$\operatorname{cis} 0 + \operatorname{cis} 5 + \operatorname{cis} 10 + \cdots + \operatorname{cis} 175 = \sum_{k=0}^{35} (\operatorname{cis} 5)^k,$$

by the theorem of DeMoivre. This is a geometric series, whose sum is

$$\frac{\operatorname{cis} 0 - \operatorname{cis} 180}{1-\operatorname{cis} 5} = \frac{2}{1-\operatorname{cis} 5} = \frac{2}{1-\cos 5 - i \sin 5} = \frac{2(1-\cos 5 + i\sin 5)}{(1-\cos 5)^2 + (\sin 5)^2}.$$

The imaginary part of this sum is

$$\frac{\sin 5}{1-\cos 5} = \frac{\sin 175}{1+\cos 175} = \tan\frac{175}{2}.$$

Thus $m+n=177$.

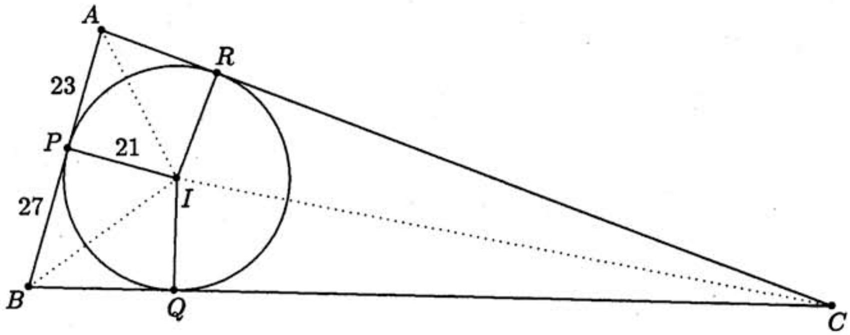
Query: Both solutions make use of the identities $\tan\frac{1}{2}x = \dfrac{\sin x}{1+\cos x} = \dfrac{1-\cos x}{\sin x}$. Can you prove them?

12. (Answer: 345)

Let I be the center of the inscribed circle, and R be the point where the circle is tangent to $\overline{CA}$. Because I is the intersection of the angle bisectors of ABC, it follows that $\angle IAB + \angle IBC + \angle ICA = 90°$. Notice that $\tan\angle IAB = 21/23$, $\tan\angle IBC = 21/27$, and $\tan\angle ICA = 21/CR$. Thus

$$\frac{CR}{21} = \tan(90° - \angle ICA) = \tan(\angle IAB + \angle IBC) = \frac{\frac{21}{23} + \frac{21}{27}}{1 - \frac{21}{23}\cdot\frac{21}{27}} = \frac{35}{6}.$$

Therefore $CR = 245/2$ and the perimeter of triangle ABC is $2\left(23 + 27 + \frac{245}{2}\right) = 345$.



OR

Use the same figure. Let $a = BC$, $b = CA$, $c = AB$, $x = CR$, and $s = \frac{1}{2}(a+b+c)$. Then $s = x + 50$, $s - a = 23$, $s - b = 27$, and $s - c = x$. The area of any triangle is the product of its inradius and its semiperimeter, so $21s = \sqrt{s(s-a)(s-b)(s-c)}$, by Heron's formula. It follows that $21^2(x+50) = 23\cdot 27\cdot x$. Solve this equation to obtain $x = 245/2$ and $2s = 2(x+50) = 345$.

OR

As in the preceding, let s be the semiperimeter of triangle ABC, and notice that $BC = s - 23$. Because $\overline{IB}$ bisects $\angle ABC$, it follows that

$$\sin\angle ABC = 2\sin\angle PBI\cos\angle PBI = 2\cdot\frac{7}{\sqrt{130}}\cdot\frac{9}{\sqrt{130}} = \frac{63}{65}.$$

Hence the area of triangle ABC is $\frac{1}{2}AB\cdot BC\sin\angle ABC = 25(s-23)\frac{63}{65}$. The area of a triangle is also equal to the product of its inradius and its semiperimeter, so $25(s-23)\frac{63}{65} = 21s$. Solve this equation to find that $2s = 345$, which is the perimeter.

13. (Answer: 742)

Suppose that no two teams win the same number of games. Then, for any k between 0 and 39, exactly one team wins k games. Moreover, a team that wins 38 games can lose only to the team that wins all of its games. An inductive argument shows that in fact each team loses to any team that wins more games, but to no other teams. Thus a tournament in which no two teams win the same number of games is uniquely determined by listing the teams in order of their wins. Given any listing, the probability that each team beat all the teams below it and lost to all the teams above it is $\left(\frac{1}{2}\right)^C$, where $C = \binom{40}{2} = 780$ is the total number of games. Because there are 40! ways to list the teams, the requested probability is $\frac{40!}{2^{780}}$. The fraction $\frac{40!}{2^{780}}$ is not in lowest terms, however. The number of factors of 2 in 40! is

$$\left\lfloor \frac{40}{2} \right\rfloor + \left\lfloor \frac{40}{2^2} \right\rfloor + \left\lfloor \frac{40}{2^3} \right\rfloor + \cdots = 20 + 10 + 5 + 2 + 1 = 38.$$

Thus the requested probability is $\frac{m}{2^{742}}$, where m is an odd integer, so $\log_2 n = 742$.

14. (Answer: 463)

Let ω denote the common measure of angles PAB, PBC, and PCA; let a, b, and c denote BC, CA, and AB; and let x, y, and z denote PA, PB, and PC. Apply the Law of Cosines to triangles PCA, PAB, and PBC to obtain

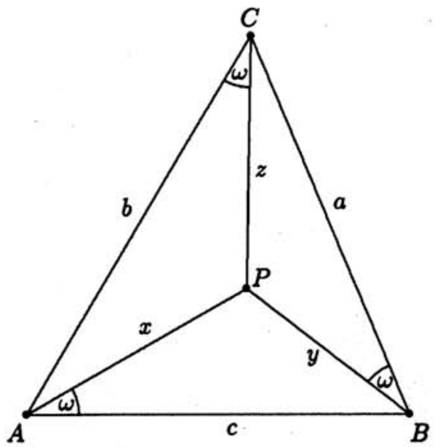
$$x^2 = z^2 + b^2 - 2bz \cos \omega$$
$$y^2 = x^2 + c^2 - 2cx \cos \omega$$
$$z^2 = y^2 + a^2 - 2ay \cos \omega.$$

Sum these three equations to obtain $2(cx + ay + bz) \cos \omega = a^2 + b^2 + c^2$. Because the combined area of triangles PAB, PBC, and PCA is $\frac{1}{2}(cx + ay + bz) \sin \omega$, the preceding equation can be rewritten as

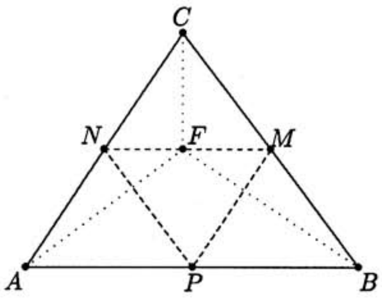
$$\tan \omega = \frac{4[ABC]}{a^2 + b^2 + c^2},$$

where $[ABC]$ denotes the area of triangle ABC. With $a = 14$, $b = 15$, and $c = 13$, use Heron's formula to find that $[ABC] = 84$. It follows that $\tan \omega = 168/295$, so $m + n = 463$.

Query: Triangle ABC has two *Brocard points*, and P is one of them. The other one is the point Q for which angles QBA, QCB, and QAC are equal. What is the common measure of these three angles?



15. (Answer: 408)



Assign coordinate triples to the vertices, so that $A = (0,0,0)$, $B = (34,0,0)$, $C = (16,24,0)$, and midpoints $M = (25,12,0)$, $N = (8,12,0)$, and $P = (17,0,0)$. Without loss of generality, assume that triangle MNP remains fixed when triangle ABC is folded. Vertex C must then stay in the plane $x = 16$ (which is perpendicular to midline $\overline{MN}$), vertex A must stay in the plane $4y = 3x$ (which is perpendicular to midline $\overline{NP}$), and vertex B must stay in the plane $2x+3y = 68$ (which is perpendicular to midline $\overline{PM}$). The intersection of these three planes includes $F = (16,12,0)$, which happens to be on $\overline{MN}$. The planes also intersect at V, the fourth vertex of the pyramid. Notice that $VF = CF = 12$. Because the planes are all perpendicular to triangle MNP, the altitude drawn to the base MNP of pyramid $VMNP$ is $\overline{VF}$. The volume of the pyramid is therefore

$$\frac{1}{3} \cdot \frac{1}{2} \cdot 17 \cdot 12 \cdot 12 = 408.$$

OR

The pyramid is a tetrahedron with four congruent acute faces. Hence its edges may be regarded as diagonals of the faces of a rectangular parallelepiped, as shown below. The edges are $d = \frac{1}{2}AB = 17$, $e = \frac{1}{2}BC = 15$, and $f = \frac{1}{2}CA = 4\sqrt{13}$. The edges of the parallelepiped are a, b, and c, where $a^2+b^2 = d^2$, $a^2+c^2 = e^2$, and $b^2+c^2 = f^2$. Solve these equations simultaneously to find that $a^2 = \frac{1}{2}(d^2+e^2-f^2) = 153$, $b^2 = \frac{1}{2}(f^2+d^2-e^2) = 136$, and $c^2 = \frac{1}{2}(e^2+f^2-d^2) = 72$. The parallelepiped consists of four congruent pyramids, each of volume $\frac{1}{6}abc$, as well as the given pyramid. Thus the volume of the given pyramid is $\frac{1}{3}abc = \frac{1}{3}\sqrt{153 \cdot 136 \cdot 72} = 408$.