AMERICAN MATHEMATICS COMPETITIONS

14th ANNUAL
AMERICAN INVITATIONAL MATHEMATICS EXAMINATION
(AIME)
THURSDAY, MARCH 28, 1996

Sponsored by

Mathematical Association of America
Society of Actuaries   Mu Alpha Theta
National Council of Teachers of Mathematics
Casualty Actuarial Society   American Statistical Association
American Mathematical Association of Two-Year Colleges
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1. DO NOT OPEN THIS BOOKLET UNTIL TOLD TO DO SO BY YOUR PROCTOR.
2. This is a 15-question, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers; i.e., there is neither partial credit nor a penalty for wrong answers.
3. No aids other than scratch paper, graph paper, ruler, compass, and protractor are permitted. In particular, **calculators are not permitted.**
4. Please print the following:

Name: ______________________________
Last   First   Middle initial

Home address: ______________________________
Street Address

____
City   State or Province   Zip or Postcode

____
Home Phone including Area Code   Gender (M or F)   Your age

____
Full Name of School   Grade Level (e.g.,11)

5. Citizenship Status: USA Citizen____ *Permanent Resident____ Other____

   If other, explain: ______________________________

   *Permanent Resident means someone seeking citizenship and currently possessing a U.S.A. Immigration "green card".
6. A combination of the AIME and AHSME scores is used to determine eligibility for participation in the U.S.A. Mathematical Olympiad (USAMO). The USAMO will be given on THURSDAY, May 2, 1996.
7. Record all your answers, and certain other information, on the AIME answer form. Only the answer form and this cover will be collected from you.

1. In a *magic square*, the sum of the three entries in any row, column, or diagonal is the same value. The figure shows four of the entries of a magic square. Find x.

x	19	96
1		

2. For each real number x, let $\lfloor x \rfloor$ denote the greatest integer that does not exceed x. For how many positive integers n is it true that $n < 1000$ and that $\lfloor \log_2 n \rfloor$ is a positive even integer?

3. Find the smallest positive integer n for which the expansion of $(xy - 3x + 7y - 21)^n$, after like terms have been collected, has at least 1996 terms.

4. A wooden cube, whose edges are one centimeter long, rests on a horizontal surface. Illuminated by a point source of light that is x centimeters directly above an upper vertex, the cube casts a shadow on the horizontal surface. The area of the shadow, which does not include the area beneath the cube, is 48 square centimeters. Find the greatest integer that does not exceed $1000x$.

5. Suppose that the roots of $x^3 + 3x^2 + 4x - 11 = 0$ are a, b, and c, and that the roots of $x^3 + rx^2 + sx + t = 0$ are $a + b$, $b + c$, and $c + a$. Find t.

6. In a five-team tournament, each team plays one game with every other team. Each team has a 50% chance of winning any game it plays. (There are no ties.) Let m/n be the probability that the tournament will produce neither an undefeated team nor a winless team, where m and n are relatively prime positive integers. Find $m + n$.

7. Two of the squares of a 7×7 checkerboard are painted yellow, and the rest are painted green. Two color schemes are equivalent if one can be obtained from the other by applying a rotation in the plane of the board. How many inequivalent color schemes are possible?

8. The *harmonic mean* of two positive numbers is the reciprocal of the arithmetic mean of their reciprocals. For how many ordered pairs of positive integers (x, y) with $x < y$ is the harmonic mean of x and y equal to 6^{20}?

9. A bored student walks down a hall that contains a row of closed lockers, numbered 1 to 1024. He opens the locker numbered 1, and then alternates between skipping and opening each closed locker thereafter. When he reaches the end of the hall, the student turns around and starts back. He opens the first closed locker he encounters, and then alternates between skipping and opening each closed locker thereafter. The student continues wandering back and forth in this manner until every locker is open. What is the number of the last locker he opens?

10. Find the smallest positive integer solution to $\tan 19x^\circ = \dfrac{\cos 96^\circ + \sin 96^\circ}{\cos 96^\circ - \sin 96^\circ}$.

11. Let P be the product of those roots of $z^6 + z^4 + z^3 + z^2 + 1 = 0$ that have positive imaginary part, and suppose that $P = r(\cos\theta^\circ + i\sin\theta^\circ)$, where $0 < r$ and $0 \le \theta < 360$. Find θ.

12. For each permutation $a_1, a_2, a_3, \ldots, a_{10}$ of the integers $1, 2, 3, \ldots, 10$, form the sum

$$|a_1 - a_2| + |a_3 - a_4| + |a_5 - a_6| + |a_7 - a_8| + |a_9 - a_{10}|.$$

The average value of all such sums can be written in the form p/q, where p and q are relatively prime positive integers. Find $p + q$.

13. In triangle ABC, $AB = \sqrt{30}$, $AC = \sqrt{6}$, and $BC = \sqrt{15}$. There is a point D for which $\overline{AD}$ bisects $\overline{BC}$ and $\angle ADB$ is a right angle. The ratio

$$\frac{\text{Area}(\triangle ADB)}{\text{Area}(\triangle ABC)}$$

can be written in the form m/n, where m and n are relatively prime positive integers. Find $m + n$.

14. A $150 \times 324 \times 375$ rectangular solid is made by gluing together $1 \times 1 \times 1$ cubes. An internal diagonal of this solid passes through the interiors of how many of the $1 \times 1 \times 1$ cubes?

15. In parallelogram $ABCD$, let O be the intersection of diagonals $\overline{AC}$ and $\overline{BD}$. Angles CAB and DBC are each twice as large as angle DBA, and angle ACB is r times as large as angle AOB. Find the greatest integer that does not exceed $1000r$.

SOLUTIONS

A 1996 Solutions Pamphlet will be sent to exam managers within a few weeks.

WRITE TO US!

Correspondence about the problems and solutions for this AIME should be addressed to:

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Comments about administrative arrangements and orders for any publications listed below should be addressed to:

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1996 USAMO

The USA Mathematical Olympiad is a 5-question, $3\frac{1}{2}$ hour, essay-type examination. The USAMO will be held on THURSDAY, May 2, 1996. Your teacher has more details on who qualifies for the USAMO in the AHSME or AIME Teachers' Manuals. The best way to prepare for the USAMO is to study the exams from previous years and to review the contents of the ARBELOS. Copies may be ordered as indicated below.

PUBLICATIONS

MINIMUM ORDER: $5 (before handling fee), US FUNDS ONLY. Canada and US orders must be prepaid. Orders are mailed 4th class, unless you specify 1st class, in which case add 20% of the total order, with a minimum of $3 and a maximum of $15. **Please note that if the correct 1st class cost is not included, the order will be sent 4th class.** Make checks payable to the American Mathematics Competitions; or give Visa or MasterCard number and expiration date.

FOREIGN ORDERS: Do **NOT** prepay; an invoice will be sent.

Examinations: Each price is for **one** copy of an exam **and** its solutions for one year. Specify the years you want and how many copies of each. All prices effective to September 1, 1996.

- **AJHSME** 1985-95, $1 per copy per year.
- **AHSME** 1983-96, $1 per copy per year.
- **AIME** 1983-96, $2 per copy per year.
- USA and International Mathematical **Olympiads** (together), 1976-95, $5 per copy per year.
- National **Summary** of Results and Awards, 1980-96, $10 per copy per year.
- Problem Book I, AHSMEs 1950-60, $8.00
- Problem Book II, AHSMEs 1961-65, $8.00
- Problem Book III, AHSMEs 1966-72, $13.50
- Problem Book IV, AHSMEs 1973-82, $13.50
- USA Mathematical Olympiad Book 1972-86, $16.00
- International Mathematical Olympiad Book I, 1959-77, $14.00
- International Mathematical Olympiad Book II, 1978-85, $11.50

The Arbelos

- Short articles and challenging problems recommended especially for AIME and USAMO qualifiers. Six volumes, $8.00 each.