The Mathematical Association of America
American Mathematics Competitions

23rd Annual

AMC 8

(American Mathematics Contest 8)

Solutions Pamphlet

Tuesday, NOVEMBER 13, 2007

This Solutions Pamphlet gives at least one solution for each problem on this year's exam and shows that all the problems can be solved using material normally associated with the mathematics curriculum for students in eighth grade or below. These solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

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1. **(D)** The first 5 weeks Theresa works a total of $8+11+7+12+10 = 48$ hours. She has promised to work $6 \times 10 = 60$ hours. She must work $60 - 48 = 12$ hours during the final week.

2. **(E)** The ratio of the number of students who preferred spaghetti to the number of students who preferred manicotti is $\dfrac{250}{100} = \dfrac{5}{2}$.

3. **(C)** The prime factorization of 250 is $2 \cdot 5 \cdot 5 \cdot 5$. The sum of 2 and 5 is 7.

4. **(D)** Georgie has 6 choices for the window in which to enter. After entering, Georgie has 5 choices for the window from which to exit. So altogether there are $6 \times 5 = 30$ different ways for Georgie to enter one window and exit another.

5. **(B)** For his birthday, Chandler gets $50 + 35 + 15 = 100$ dollars. Therefore, he needs $500 - 100 = 400$ dollars more. It will take Chandler $400 \div 16 = 25$ weeks to earn 400 dollars, so he can buy his bike after 25 weeks.

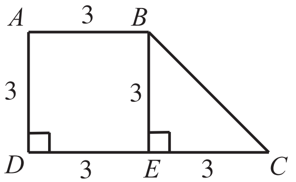
6. **(E)** The difference in the cost of a long-distance call per minute from 1985 to 2005 was $41 - 7 = 34$ cents. The percent decrease is $100 \times \dfrac{34}{41} \approx 100 \times \dfrac{32}{40} = 100 \times \dfrac{8}{10} = 80\%$.

7. **(D)** Originally the sum of the ages of the people in the room is $5 \times 30 = 150$. After the 18-year-old leaves, the sum of the ages of the remaining people is $150 - 18 = 132$. So the average age of the four remaining people is $\frac{132}{4} = 33$ years.

   OR

   The 18-year-old is 12 years younger than 30, so the four remaining people are an average of $\frac{12}{4} = 3$ years older than 30.

8. **(B)** Note that $ABED$ is a square with side 3. Subtract DE from DC, to find that $\overline{EC}$, the base of $\triangle BEC$, has length 3. The area of $\triangle BEC$ is $\dfrac{1}{2} \cdot 3 \cdot 3 = \dfrac{9}{2} = 4.5$.



   OR

   The area of the $\triangle BEC$ is the area of the trapezoid $ABCD$ minus the area of the square $ABED$. The area of $\triangle BEC$ is $\frac{1}{2}(3+6)3 - 3^2 = 13.5 - 9 = 4.5$.

9. **(B)** The number in the last column of the second row must be 1 because there are already a 2 and a 3 in the second row and a 4 in the last column. By similar reasoning, the number above the 1 must be 3. So the number in the lower right-hand square must be 2. This is not the only way to find the solution.

1		2	3
2	3		1
			4
			2

The completed square is

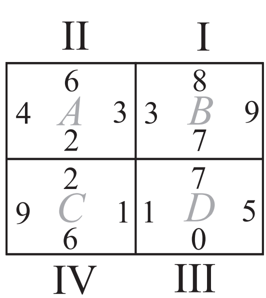
1	4	2	3
2	3	4	1
3	2	1	4
4	1	3	2

10. **(D)** First calculate $\boxed{11} = 1 + 11 = 12$. So

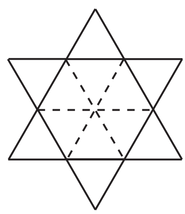
$$\boxed{\boxed{11}} = \boxed{12} = 1 + 2 + 3 + 4 + 6 + 12 = 28$$

.

11. **(D)** Because Tile III has a 0 on the bottom edge and there is no 0 on any other tile, Tile III must be placed on C or D. Because Tile III has a 5 on the right edge and there is no 5 on any other tile, Tile III must be placed on the right, on D. Because Tile III has a 1 on the left edge and only Tile IV has a 1 on the right edge, Tile IV must be placed to the left of Tile III, that is, on C.



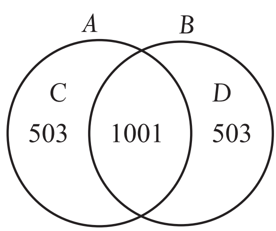
12. **(A)** Use diagonals to cut the hexagon into 6 congruent triangles. Because each exterior triangle is also equilateral and shares an edge with an internal triangle, each exterior triangle is congruent to each interior triangle. Therefore, the ratio of the area of the extensions to the area of the hexagon is 1:1.



13. **(C)** Let C denote the set of elements that are in A but not in B. Let D denote the set of elements that are in B but not in A. Because sets A and B have the same number of elements, the number of elements in C is the same as the number of elements in D. This number is half the number of elements in the union of A and B minus the intersection of A and B. That is, the number of elements in each of C and D is

$$\frac{1}{2}(2007 - 1001) = \frac{1}{2} \cdot 1006 = 503.$$

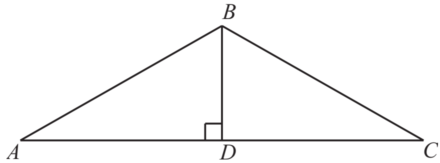
Adding the number of elements in A and B to the number in A but not in B gives $1001 + 503 = 1504$ elements in A.



OR

Let x be the number of elements each in A and B. Then $2x - 1001 = 2007$, $2x = 3008$ and $x = 1504$.

14. **(C)** Let $\overline{BD}$ be the altitude from B to $\overline{AC}$ in $\triangle ABC$.



Then $60 =$ the area of $\triangle ABC = \frac{1}{2} \cdot 24 \cdot BD$, so $BD = 5$. Because $\triangle ABC$ is isosceles, $\triangle ABD$ and $\triangle CBD$ are congruent right triangles. This means that $AD = DC = \dfrac{24}{2} = 12$. Applying the Pythagorean Theorem to $\triangle ABD$ gives

$$AB^2 = 5^2 + 12^2 = 169 = 13^2, \text{ so } AB = 13.$$

15. **(A)** Because $b < c$ and $0 < a$, adding corresponding sides of the inequalities gives $b < a + c$, so (A) is impossible. To see that the other choices are possible, consider the following choices for a, b, and c:

(B) and (C): $a = 1$, $b = 2$, and $c = 4$;

(D): $a = \dfrac{1}{3}$, $b = 1$, and $c = 2$;

(E): $a = \dfrac{1}{2}$, $b = 1$, and $c = 2$.

16. **(A)** The circumferences of circles with radii 1 through 5 are 2π, 4π, 6π, 8π and 10π, respectively. Their areas are, respectively, $\pi, 4\pi$, 9π, 16π and 25π. The points $(2\pi, \pi)$, $(4\pi, 4\pi)$, $(6\pi, 9\pi)$, $(8\pi, 16\pi)$ and $(10\pi, 25\pi)$ are graphed in **(A)**. It is the only graph of an increasing quadratic function, called a parabola.

17. **(C)** There are $0.30(30) = 9$ liters of yellow tint in the original 30-liter mixture. After adding 5 liters of yellow tint, 14 of the 35 liters of the new mixture are yellow tint. The percent of yellow tint in the new mixture is $100 \times \dfrac{14}{35} = 100 \times \dfrac{2}{5}$ or 40%.

18. **(D)** To find A and B, it is sufficient to consider only $303 \cdot 505$, because 0 is in the thousands place in both factors.

$$\begin{array}{r} \cdots 303 \\ \times \quad \cdots 505 \\ \hline \cdots 1515 \\ \cdots 1500 \\ \hline \cdots 3015 \end{array}$$

So $A = 3$ and $B = 5$, and the sum is $A + B = 3 + 5 = 8$.

19. **(C)** One of the squares of two consecutive integers is odd and the other is even, so their difference must be odd. This eliminates A, B and D. The largest consecutive integers that have a sum less than 100 are 49 and 50, whose squares are 2401 and 2500, with a difference of 99. Because the difference of the squares of consecutive positive integers increases as the integers increase, the difference cannot be 131. The difference between the squares of 40 and 39 is 79.

OR

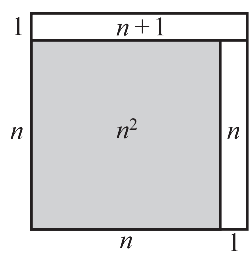
Let the consecutive integers be n and $n + 1$, with $n \le 49$. Then

$$(n + 1)^2 - n^2 = (n^2 + 2n + 1) - n^2 = 2n + 1 = n + (n + 1).$$

That means the difference of the squares is an odd number. Therefore, the difference is an odd number less than or equal to $49 + (49 + 1) = 99$, and choice C is the only possible answer. The sum of $n = 39$ and $n + 1 = 40$ is 79.

Note: The difference of the squares of any two consecutive integers is not only odd but also the sum of the two consecutive integers. Every positive odd integer greater than 1 and less than 100 could be the answer.

Seen in geometric terms, $(n + 1)^2 - n^2$ looks like



20. **(A)** Because 45% is the same as the simplified fraction $\frac{9}{20}$, the Unicorns won 9 games for each 20 games they played. This means that the Unicorns must have played some multiple of 20 games before district play. The table shows the possibilities that satisfy the conditions in the problem.

Before District Play			**After District Play**		
Games Played	Games Won	Games Lost	Games Played	Games Won	Games Lost
20	9	11	28	15	13
40	**18**	**22**	**48**	**24**	**24**
60	27	33	68	33	35
80	36	44	88	42	46
...	...	...	...	...	...

Only when the Unicorns played 40 games before district play do they finish winning half of their games. So the Unicorns played $24 + 24 = 48$ games.

OR

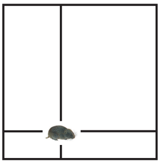
Let n be the number of Unicorn games before district play. Then $0.45n + 6 = 0.5(n + 8)$. Solving for n yields

$$\begin{aligned} 0.45n + 6 &= 0.5n + 4, \\ 2 &= 0.05n, \\ 40 &= n. \end{aligned}$$

So the total number of games is $40 + 8 = 48$.

21. **(D)** After the first card is dealt, there are seven left. The three cards with the same color as the initial card are winners and so is the card with the same letter but a different color. That means four of the remaining seven cards form winning pairs with the first card, so the probability of winning is $\frac{4}{7}$.

22. **(C)** Wherever the lemming is inside the square, the sum of the distances to the two horizontal sides is 10 meters and the sum of the distances to the two vertical sides is 10 meters. Therefore the sum of all four distances is 20 meters, and the average of the four distances is $\dfrac{20}{4} = 5$ meters.



23. **(B)** Find the area of the unshaded portion of the 5×5 grid, then subtract the unshaded area from the total area of the grid. The unshaded triangle in the middle of the top of the 5×5 grid has a base of 3 and an altitude of $\dfrac{5}{2}$. The four unshaded triangles have a total area of $4 \times \dfrac{1}{2} \times 3 \times \dfrac{5}{2} = 15$ square units. The four corner squares are also unshaded, so the shaded pinwheel has an area of $25 - 15 - 4 = 6$ square units.

24. **(C)** A number is a multiple of three when the sum of its digits is a multiple of 3. If the number has three distinct digits drawn from the set $\{1, 2, 3, 4\}$, then the sum of the digits will be a multiple of three when the digits are $\{1, 2, 3\}$ or $\{2, 3, 4\}$. That means the number formed is a multiple of three when, after the

three draws, the number remaining in the bag is 1 or 4. The probability of this occurring is $\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$.

25. **(B)** The outer circle has area 36π and the inner circle has area 9π, making the area of the outer ring $36\pi - 9\pi = 27\pi$. So each region in the outer ring has area $\frac{27\pi}{3} = 9\pi$, and each region in the inner circle has area $\frac{9\pi}{3} = 3\pi$. The probability of hitting a given region in the inner circle is $\frac{3\pi}{36\pi} = \frac{1}{12}$, and the probability of hitting a given region in the outer ring is $\frac{9\pi}{36\pi} = \frac{1}{4}$. For the score to be odd, one of the numbers must be 1 and the other number must be 2. The probability of hitting a 1 is

$$\frac{1}{4}+\frac{1}{4}+\frac{1}{12}=\frac{7}{12},$$

and the probability of hitting a 2 is

$$1-\frac{7}{12}=\frac{5}{12}.$$

Therefore, the probability of hitting a 1 and a 2 in either order is

$$\frac{7}{12}\cdot\frac{5}{12}+\frac{5}{12}\cdot\frac{7}{12}=\frac{70}{144}=\frac{35}{72}.$$

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