AMERICAN MATHEMATICS COMPETITIONS

10th ANNUAL
AMERICAN INVITATIONAL MATHEMATICS EXAMINATION
(AIME)
THURSDAY, APRIL 2, 1992

Sponsored by

Mathematical Association of America
Society of Actuaries   Mu Alpha Theta
National Council of Teachers of Mathematics
Casualty Actuarial Society   American Statistical Association
American Mathematical Association of Two-Year Colleges
American Mathematical Society

1. DO NOT OPEN THIS BOOKLET UNTIL TOLD TO DO SO BY YOUR PROCTOR.
2. This is a 15-question, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers; i.e., there is neither partial credit nor a penalty for wrong answers.
3. Scratch paper, graph paper, ruler, compass, protractor and eraser are permitted. *Calculators and slide rules are not permitted.*
4. Please print the following:

Name: ______________________________
Last   First   Middle initial

Home address: ______________________________
Street Address

____
City   State or Province   Zip or Postcode

____
Home Phone including Area Code   Sex (M or F)   Your age

____
Full Name of School   Grade Level (e.g. 11)

5. Citizenship Status: USA Citizen_____ Permanent Resident_____ Other_____
   If other, explain: ______________________________
6. My score on the 1992 AHSME was [   ]   I took the 1992 AHSME on (date): ______________
7. A combination of the AIME and AHSME scores is used to determine eligibility for participation in the U. S. A. Mathematical Olympiad (USAMO). The USAMO will be given on THURSDAY April 30, 1992. Please check one box:

   If I qualify for the USAMO, I agree to take it.   YES ☐   NO ☐

   (*Your school must also agree to administer the USAMO before you can take it.*)
8. Record all your answers, and certain other information, on the AIME answer form. Your Examination Manager will instruct you how to complete the form. Only the answer form and this cover will be collected from you.

1. Find the sum of all positive rational numbers that are less than 10 and that have denominator 30 when written in lowest terms.

2. A positive integer is called "ascending" if, in its decimal representation, there are at least two digits and each digit is less than any digit to its right. How many ascending positive integers are there?

3. A tennis player computes her "win ratio" by dividing the number of matches she has won by the total number of matches she has played. At the start of a weekend, her win ratio is exactly .500. During the weekend she plays four matches, winning three and losing one. At the end of the weekend her win ratio is greater than .503. What is the largest number of matches that she could have won before the weekend began?

4. In Pascal's triangle, each entry is the sum of the two entries above it. The first few rows of the triangle are shown below.

Row 0:							1						
Row 1:						1		1					
Row 2:					1		2		1				
Row 3:				1		3		3		1			
Row 4:			1		4		6		4		1		
Row 5:		1		5		10		10		5		1	
Row 6:	1		6		15		20		15		6		1

In which row of Pascal's triangle do three consecutive entries occur that are in the ratio $3:4:5$?

5. Let S be the set of all rational numbers r, $0 < r < 1$, that have a repeating decimal expansion of the form

$$0.abcabcabc\ldots = 0.\overline{abc},$$

where the digits a, b, c are not necessarily distinct. To write the elements of S as fractions in lowest terms, how many different numerators are required?

6. For how many pairs of consecutive integers in $\{1000, 1001, 1002, \ldots, 2000\}$ is no carrying required when the two integers are added?

7. Faces ABC and BCD of tetrahedron $ABCD$ meet at an angle of 30°. The area of face ABC is 120, the area of face BCD is 80, and $BC = 10$. Find the volume of the tetrahedron.

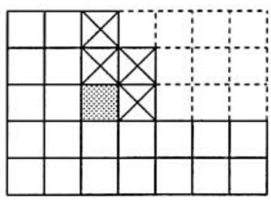
8. For any sequence of real numbers $A = (a_1, a_2, a_3, \ldots)$, define ΔA to be the sequence $(a_2 - a_1, a_3 - a_2, a_4 - a_3, \ldots)$, whose n^{th} term is $a_{n+1} - a_n$. Suppose that all of the terms of the sequence $\Delta(\Delta A)$ are 1, and that $a_{19} = a_{92} = 0$. Find a_1.

9. Trapezoid $ABCD$ has sides $AB = 92$, $BC = 50$, $CD = 19$, and $AD = 70$, with $\overline{AB}$ parallel to $\overline{CD}$. A circle with center P on $\overline{AB}$ is drawn tangent to $\overline{BC}$ and $\overline{AD}$. Given that $AP = m/n$, where m and n are relatively prime positive integers, find $m + n$.

10. Consider the region A in the complex plane that consists of all points z such that both $z/40$ and $40/\overline{z}$ have real and imaginary parts between 0 and 1 inclusive. What is the integer that is nearest the area of A? (If $z = x + iy$ with x and y real, then $\overline{z} = x - iy$ is the conjugate of z.)

11. Lines ℓ_1 and ℓ_2 both pass through the origin and make first-quadrant angles of $\dfrac{\pi}{70}$ and $\dfrac{\pi}{54}$ radians, respectively, with the positive x-axis. For any line ℓ, the transformation $R(\ell)$ produces another line as follows: ℓ is reflected in ℓ_1, and the resulting line is then reflected in ℓ_2. Let $R^{(1)}(\ell) = R(\ell)$, and for integer $n \geq 2$ define $R^{(n)}(\ell) = R\left(R^{(n-1)}(\ell)\right)$. Given that ℓ is the line $y = \dfrac{19}{92}x$, find the smallest positive integer m for which $R^{(m)}(\ell) = \ell$.

12. In a game of *Chomp*, two players alternately take "bites" from a 5-by-7 grid of unit squares. To take a bite, the player chooses one of the remaining squares, then removes ("eats") all squares found in the quadrant defined by the left edge (extended upward) and the lower edge (extended rightward) of the chosen square. For example, the bite determined by the shaded square in the diagram would remove the shaded square and the four squares marked by $\times$. (The squares with two or more dotted edges have been removed from the original board in previous moves.) The object of the game is to make one's opponent take the last bite. The diagram shows one of the many subsets of the set of 35 unit squares that can occur during games of Chomp. How many different subsets are there in all? Include the full board and the empty board in your count.



13. Triangle ABC has $AB = 9$ and $BC : CA = 40 : 41$. What is the largest area that this triangle can have?

14. In triangle ABC, A', B', and C' are on sides $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$, respectively. Given that $\overline{AA'}$, $\overline{BB'}$, and $\overline{CC'}$ are concurrent at the point O, and that

$$\frac{AO}{OA'} + \frac{BO}{OB'} + \frac{CO}{OC'} = 92,$$

find the value of

$$\frac{AO}{OA'} \cdot \frac{BO}{OB'} \cdot \frac{CO}{OC'}.$$

15. Define a positive integer n to be a "factorial tail" if there is some positive integer m such that the base-ten representation of $m!$ ends with exactly n zeros. How many positive integers less than 1992 are *not* factorial tails?

SOLUTIONS

A 1992 Solutions Pamphlet will be sent to exam managers within a few weeks.

WRITE TO US!

Questions and comments about the problems and solutions for this AIME (but **not** requests for the Solutions Pamphlet) should be addressed to:

Prof Elgin H Johnston, AIME Chairman
Department of Mathematics
Iowa State University, Ames, IA 50011 USA

Comments about administrative arrangements and orders for any publications listed below should be addressed to:

Prof Walter E Mientka, AMC Executive Director
Department of Mathematics and Statistics
University of Nebraska, Lincoln, NE 68588-0658 USA

1992 USAMO

The USA Mathematical Olympiad is a 5-question, $3\frac{1}{2}$ hour, essay-type examination. The USAMO will be held on THURSDAY April 30, 1992. Your teacher has more details on who qualifies for the USAMO in the AHSME or AIME Teachers' Manuals. The best way to prepare for the USAMO is to study the exams from previous years and to review the contents of the ARBELOS. Copies may be ordered as indicated below.

PUBLICATIONS

MINIMUM ORDER: $5 (before handling fee), US FUNDS ONLY. Canada and US orders must be prepaid. Orders are mailed 4th class, unless you specify 1st class, in which case add $3.00 or 20% of total order, whichever is larger, with a maximum of $15.00. Make checks payable to the American Mathematics Competitions.

FOREIGN ORDERS: do **NOT** prepay; an invoice will be sent.

Examinations: Each price is for **one** copy of an exam **and** its solutions for one year. Specify the years you want and how many copies of each. All prices effective to September 1, 1992.

- **AJHSME** (Junior High Exam), 1985-1991, $1 per copy per year.
- **AHSME** 1980-92, $1 per copy per year.
- **AIME** 1983-92, $2 per copy per year.
- USA and International Mathematical **Olympiads** (together), 1976-91, $3 per copy per year.
- National **Summary** of Results and Awards, 1980-91, $4 per copy per year.

Books (Exams and solutions):
- Contest Problem Book I, AHSMEs 1950-60, $8.50.
- Contest Problem Book II, AHSMEs 1961-65, $8.50.
- Contest Problem Book III, AHSMEs 1966-72, $10.00.
- Contest Problem Book IV, AHSMEs 1973-82, $11.00.
- USA Mathematical Olympiads, 1972-86, $13.00.
- International Mathematical Olympiads, 1959-77, $10.00.
- International Mathematical Olympiads, 1978-85, $11.00.

Journal
- The ARBELOS (short articles and challenging problems); recommended especially for AIME and USAMO qualifiers. Five volumes plus a Geometry volume, $7.00 each.