AMERICAN MATHEMATICS COMPETITIONS

# AIME SOLUTIONS PAMPHLET
## FOR STUDENTS AND TEACHERS

## 15th ANNUAL
# AMERICAN INVITATIONAL MATHEMATICS EXAMINATION
# (AIME)
## THURSDAY, March 20, 1997

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This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs. geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution. *Remember that reproduction of these solutions is prohibited by copyright.*

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1. (Answer: 750)
A difference of integer squares $a^2 - b^2$ can be factored as $(a+b)(a-b)$, the product of two integers of the same parity. Thus, for an even integer to be a difference of integer squares, it must be a product of two even integers, and therefore must be divisible by 4. Conversely, any integer multiple of 4 may be expressed as the difference of two integer squares, because $4n = (n+1)^2 - (n-1)^2$. Any odd number may be expressed as the difference of two integer squares, because $2n+1 = (n+1)^2 - n^2$. In all, exactly 750 of the integers from 1 to 1000 are differences of integer squares.

2. (Answer: 125)
Notice that rectangles must be formed by choosing two distinct vertical lines and two distinct horizontal lines. Because there are nine vertical lines and nine horizontal lines, the total number of rectangles is $\binom{9}{2}^2 = 1296$. Of these, 64 are $1 \times 1$ squares, 49 are $2 \times 2$ squares, and, in general, $(9-j)^2$ are $j \times j$ squares, because each square is determined by its size and the position of its upper-left corner. Hence the total number of squares on the $8 \times 8$ checkerboard is $64+49+36+25+16+9+4+1 = 204$. Thus
$$\frac{s}{r} = \frac{204}{1296} = \frac{17}{108},$$
and $m+n = 125$.

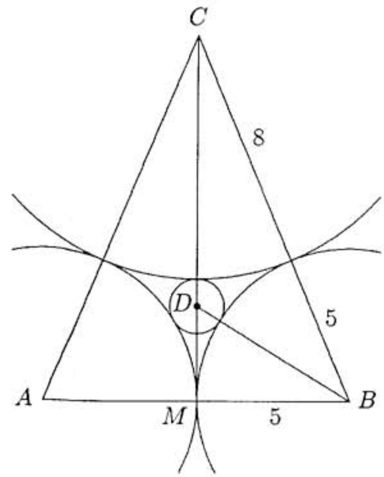
3. (Answer: 126)
Let $x$ and $y$ be the two- and three-digit numbers, respectively. It is given that the juxtaposition is nine times as large as the intended product, hence
$$1000x + y = 9xy, \text{ or}$$
$$\frac{y}{9y-1000} = x.$$
Because $9y - 1000$ must be positive, the smallest possible value for $y$ is 112. Because $x$ must be at least 10, it follows that $10(9y-1000) \leq y$, hence that $y \leq \frac{10000}{89} < 113$. Thus $y$ can only be 112, and $x$ must be 14, so $x + y = 126$.

4. (Answer: 017)

Let $A$, $B$, $C$, and $D$ be the centers of the circles of radii 5, 5, 8, and $r = m/n$, respectively. It is clear that $C$ lies outside the strip bounded by the parallel lines that are tangent to both of the circles of radius 5. Thus $D$ lies inside triangle $ABC$, as in the figure at right. Notice that, because the centers of two tangent circles are collinear with their point of tangency, $AB = 10$, $CA = CB = 13$, and $DB = 5 + r$. Because $DA = DB$, $D$ is on the perpendicular bisector $\overline{MC}$ of $\overline{AB}$, where $M$ is the midpoint of $\overline{AB}$. Apply the Pythagorean Theorem to triangle $CMB$ to find that $CM = 12$, hence that $DM = CM - CD = 12 - (8 + r) = 4 - r$. Now apply the Pythagorean Theorem again, this time to triangle $DMB$, to find that

$$(5 + r)^2 = 5^2 + (4 - r)^2.$$

The solution to this equation is $r = \dfrac{8}{9}$, hence $m + n = 8 + 9 = 17$.




**Query.** Given three mutually externally tangent circles, whose radii are $r_1 \le r_2 \le r_3$, under what conditions will there be more than one circle that is externally tangent to all three circles?

5. (Answer: 417)

Let $S$ be the set of real numbers that can be written as fractions whose numerators are 1 or 2 and whose denominators are integers. Because $\frac{1}{4}$ is the greatest element of $S$ that is less than $\frac{2}{7}$, and $\frac{1}{3}$ is the least element of $S$ that is greater than $\frac{2}{7}$, we must find the number of values for $r$ that are closer to $\frac{2}{7}$ than to $\frac{1}{3}$ or $\frac{1}{4}$. Because $r$ can be expressed as a four-place decimal, the inequality

$$\frac{\frac{1}{4} + \frac{2}{7}}{2} < r < \frac{\frac{2}{7} + \frac{1}{3}}{2}$$

implies that $0.2679 \le r \le 0.3095$. Thus there are $3095 - 2679 + 1 = 417$ possible values for $r$.

**Query**. The length of the interval $\frac{15}{56} < r < \frac{13}{42}$ is $\frac{1}{24}$. How many four-place decimals might there be in an arbitrary interval of length $\frac{1}{24}$?

6. (Answer: 042)
Let $m$ be the number of sides of the polygon determined by $A_n$, $A_1$, and $B$. The degree measures of the interior angles of the three polygons are $180 - \frac{360}{n}$, 60, and $180 - \frac{360}{m}$. If $6 < n$, the polygons fit together at their common vertex $A_1$, thus

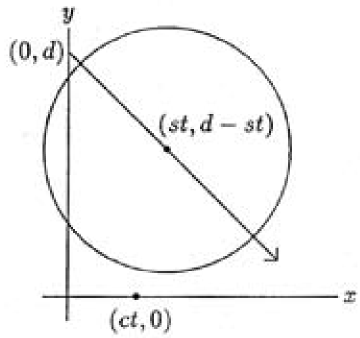
$$360 = 180 - \frac{360}{n} + 60 + 180 - \frac{360}{m}.$$

This can be rewritten in the form

$$n = \frac{6m}{m-6} = 6 + \frac{36}{m-6}.$$

It is clear that $m > 6$, so that $n$ is a decreasing function of $m$. The largest value of $n$ is 42, obtained when $m = 7$.

7. (Answer: 198)
Let $r$ be the radius of the storm, $d$ be the distance from the center of the storm to the car at time $t = 0$, $c$ be the speed of the car, and $s\sqrt{2}$ be the speed of the storm. Set up a coordinate system so that, at $t = 0$, the storm center is at $(0, d)$, and the car is at the origin, moving along the positive $x$-axis. At any time $t$, the car is at $(ct, 0)$ and the storm center is at $(st, d - st)$. When the car is entering or leaving the storm circle, the distance between these two points is $r$. In other words, $t_1$ and $t_2$ are the solutions to




$$(ct - st)^2 + (d - st)^2 = r^2,$$

which can be rewritten as $((c-s)^2 + s^2)t^2 - (2ds)t + (d^2 - r^2) = 0$. The sum of the two roots of this quadratic equation is

$$t_1 + t_2 = \frac{2ds}{(c-s)^2 + s^2}.$$

Now use the given data $c = \frac{2}{3}$, $d = 110$, and $s = \frac{1}{2}$ to find that

$$\frac{t_1 + t_2}{2} = \frac{ds}{(c-s)^2 + s^2} = \frac{110 \cdot \frac{1}{2}}{\left(\frac{2}{3} - \frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = 198.$$

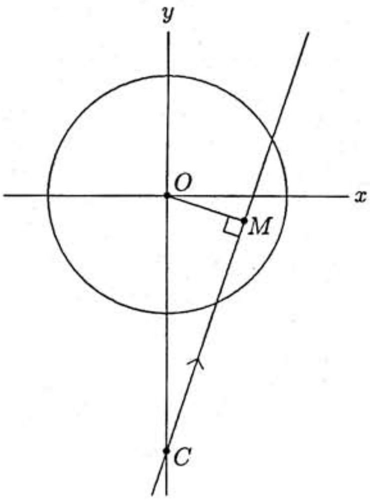
The discriminant $4d^2s^2 - 4(d^2 - r^2)((c-s)^2 + s^2)$ of the quadratic equation is nonnegative in this case, therefore $t_1$ and $t_2$ are real. The answer is otherwise independent of $r$.

**OR**

Make the storm center the origin $O$ of a coordinate system that has its positive $x$-axis pointing east. At time $t = 0$, the car is at $C = (0, -110)$. Each minute thereafter, its position relative to the storm center is shifted by the velocity vector

$$(2/3, 0) - (1/2, -1/2) = (1/6, 1/2),$$

due to the combined motion of the car and storm, respectively. The resulting linear path of the car intersects the storm along a chord of the circle, whose midpoint $M$ is reached at time $t = \frac{1}{2}(t_1 + t_2)$. Because angle $CMO$ is right, and $\tan \angle OCM$ is $\frac{1}{3}$, it follows that $CM = CO \cos \angle OCM = 110 \cdot \frac{3}{\sqrt{10}} = 33\sqrt{10}$. To find the time it takes for the car to reach $M$, divide this distance by the relative speed of the car, which is $\frac{1}{6}\sqrt{10}$. It follows that




$$\frac{1}{2}(t_1 + t_2) = \frac{33\sqrt{10}}{\frac{1}{6}\sqrt{10}} = 198.$$

8. (Answer: 090)
Each row and each column must contain two 1's and two $-1$'s, so there are $\binom{4}{2} = 6$ ways to fill the first row. There are also six ways to fill the second row. Of these, one way has four matches with the first row, four ways have two matches with the first row, and one way has no matches with the first row. The first case allows one way to fill the third row, the second case allows two ways to fill the third row, and the third case allows six ways to fill the third row. Once the first three rows are filled, the fourth row can be filled in only one way. There are thus $6(1 \cdot 1 + 4 \cdot 2 + 1 \cdot 6) = 90$ ways to fill the array to satisfy the conditions.

9. (Answer: 233)
Notice first that the given data imply that $\langle a^{-1} \rangle = a^{-1}$ and $\langle a^2 \rangle = a^2 - 2$. Hence $a$ must satisfy the equation $a^{-1} = a^2 - 2$, or $a^3 - 2a - 1 = 0$. This factors as

$$(a+1)(a^2 - a - 1) = 0,$$

whose only positive root is $a = \frac{1}{2}(1+\sqrt{5})$. Now use the relations $a^2 = a + 1$ and $a^3 = 2a + 1$ to calculate

$$a^6 = 8a + 5,$$
$$a^{12} = 144a + 89,$$
$$a^{13} = 233a + 144,$$

from which it follows that $a^{12} - 144a^{-1} = \dfrac{a^{13} - 144}{a} = 233$.

**OR**

As above, show that $a = \frac{1}{2}(1+\sqrt{5})$. The cubic equation yields $a^3 = 2a + 1 = 2 + \sqrt{5}$. It follows that $a^6 = 9 + 4\sqrt{5}$ and $a^{12} = 161 + 72\sqrt{5}$. The cubic equation also implies that $144a^{-1} = 144(a^2 - 2) = -72 + 72\sqrt{5}$, hence $a^{12} - 144a^{-1} = 233$.

**OR**

As above, $a = (1+\sqrt{5})/2$. Let $b = -a^{-1}$ and use Binet's formula for Fibonacci numbers to calculate

$$F_n = \frac{a^n - b^n}{\sqrt{5}} = \frac{a^n - b^n}{a - b} = a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}.$$

Thus

$$F_n = a^{n-1} + F_{n-1}b = a^{n-1} - F_{n-1}a^{-1}.$$

In particular,

$$233 = F_{13} = a^{12} - F_{12}a^{-1} = a^{12} - 144a^{-1}.$$

10. (Answer: 117)
Consider any pair of cards from the deck. We show that there is exactly one card that can be added to this pair to make a complementary set. If the cards in the pair have the same shape, then the third card must also have this shape, while if the cards have different shapes, then the third card must have the one shape that differs from them. In either case, the shape on the third card is uniquely determined. Similar reasoning shows that the color and the shade on the third card are also uniquely determined. Thus we can count the number of complementary sets by counting the number of pairs of cards and then dividing by 3, because each complementary set is counted three times by this procedure. The number of complementary sets is

$$\frac{1}{3}\binom{27}{2} = \frac{1}{3} \cdot \frac{27 \cdot 26}{2} = 117.$$

11. (Answer: 241)
Because $\sin n^\circ + \cos n^\circ = \sqrt{2}\cos(45-n)^\circ$, it follows that

$$\sum_{n=1}^{44} \sin n^\circ + \sum_{n=1}^{44} \cos n^\circ = \sum_{n=1}^{44} \sqrt{2}\cos(45-n)^\circ = \sum_{n=1}^{44} \sqrt{2}\cos n^\circ.$$

Thus

$$\sum_{n=1}^{44} \sin n^\circ = (\sqrt{2}-1)\sum_{n=1}^{44} \cos n^\circ,$$

which yields $x = 1+\sqrt{2}$ and $\lfloor 100x \rfloor = 241$.

12. (Answer: 058)

The statement implies that $f$ is its own inverse. The inverse may be found by solving $x = \dfrac{ay+b}{cy+d}$ for $y$. This yields $f^{-1}(x) = \dfrac{dx-b}{-cx+a}$. Because the nonzero numbers $a$, $b$, $c$, and $d$ must therefore be proportional to $-d$, $b$, $c$, and $-a$, respectively, it follows that $a=-d$, hence that $f(x) = \dfrac{ax+b}{cx-a}$. The conditions $f(19)=19$ and $f(97)=97$ lead to the equations

$$19^2c = 2\cdot 19a + b$$
$$97^2c = 2\cdot 97a + b.$$

Thus $(97^2-19^2)c = 2(97-19)a$, from which follows $a = 58c$, which in turn leads to $b=-1843c$. This determines

$$f(x) = \frac{58x-1843}{x-58} = 58 + \frac{1521}{x-58},$$

which never has the value 58.

**OR**

The number that is not in the domain of $f$ is $x=-d/c$, and the number that is not in the range of $f$ is $y = a/c$. Because $f = f^{-1}$, it follows that $d=-a$. The fixed points of $f$ satisfy

$$ax+b = x(cx+d), \qquad \text{or} \qquad cx^2-(a-d)x-b=0.$$

It follows that the sum of the two fixed points is $(a-d)/c = 2a/c$, which is twice the number omitted from the range. Because the fixed points are given as 19 and 97, the number omitted from the range is (19+97)/2=58.

**OR**

The equation $y = \dfrac{ax+b}{cx+d}$ is equivalent to the equation $xy - \dfrac{a}{c}x + \dfrac{d}{c}y = \dfrac{b}{c}$, hence to the equation $\left(x+\dfrac{d}{c}\right)\left(y-\dfrac{a}{c}\right) = \dfrac{bc-ad}{c^2}$. Notice that $bc-ad$ cannot be zero, for this would imply that $f(x) = \dfrac{a}{c}$ for all values of $x$ except $-\dfrac{d}{c}$, contrary to the problem statement. The graph of $y=f(x)$ is therefore a hyperbola, whose center is $\left(\dfrac{-d}{c}, \dfrac{a}{c}\right)$. The statement implies that the line $y=x$ is an axis of symmetry. Because the hyperbola intersects this axis at (19,19) and at (97,97), these points are the vertices of the hyperbola, and its center is midway between them. It follows that $\dfrac{-d}{c} = \dfrac{a}{c} = \dfrac{19+97}{2} = 58$, which is the value omitted from the range of $f$ (and also from the domain of $f$).

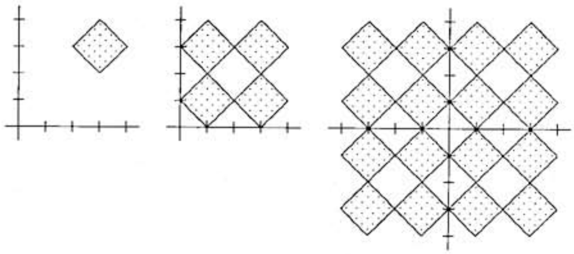
13. (Answer: 066)
Making use of symmetry, graph the part in the first quadrant and then reflect this in the coordinate axes. In the first quadrant, the defining equation simplifies to

$$\Big||x-2|-1\Big|+\Big||y-2|-1\Big|=1.$$

Again making use of symmetry, graph the part in the region $2 \le x$, $2 \le y$ and then reflect this in the lines $x = 2$ and $y = 2$. In this region, the equation simplifies further to

$$|x-3|+|y-3|=1,$$

the graph of which is a square, whose vertices are (3,2), (4,3), (3,4), and (2,3), and whose perimeter is $4\sqrt{2}$.



Reflection in the line $x = 2$ and then in the line $y = 2$ produces a set of squares for which the required length of wire is $4 \cdot 4\sqrt{2} = 16\sqrt{2}$, as shown in the middle figure. Reflection in the coordinate axes then produces a set of squares for which the required length of wire is $4 \cdot 16\sqrt{2} = 64\sqrt{2}$, as shown in the third figure. Thus $a + b = 66$.

14. (Answer: 582)
Because the 1997 roots of the equation are symmetrically distributed in the complex plane, it is no loss of generality to assume that $v = 1$. Let $w = \cos\theta + i\sin\theta$, with $-180° < \theta < 180°$. It is required to find the probability that

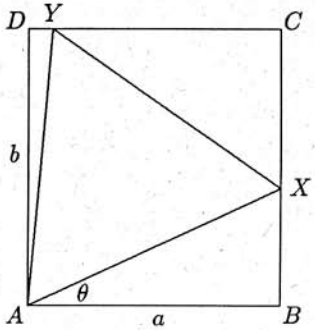
$$|1+w|^2 = |(1+\cos\theta)+i\sin\theta|^2 = 2+2\cos\theta \ge 2+\sqrt{3},$$

which is equivalent to $\cos\theta \ge \frac{1}{2}\sqrt{3}$. Thus $|\theta| \le 30°$. Because $w \ne 1$, the only possible values of $\theta$ are

$$\theta = \pm\frac{360°}{1997}\,,\ \pm\frac{720°}{1997}\,,\ \pm\frac{1080°}{1997}\,,\ldots,\ \pm\frac{360k°}{1997}\,,$$

where $k = \lfloor 1997/12 \rfloor = 166$. Hence the probability is $2 \cdot 166/1996 = 83/499$, and $m + n = 83 + 499 = 582$.

15. (Answer: 554)




Let us generalize the problem slightly. Label the rectangle $\mathcal{R} = ABCD$ so that $AB = CD = a$ and $BC = DA = b$. We first show that, given any equilateral triangle $\mathcal{T}$ in $\mathcal{R}$, there is an equilateral triangle in $\mathcal{R}$ that has the same area as $\mathcal{T}$ and that has one vertex at a vertex of $\mathcal{R}$. Given a side of $\mathcal{R}$, find the vertex of $\mathcal{T}$ that is closest to the side, then draw a line through the vertex parallel to that side. When this is done for all four sides of $\mathcal{R}$, we have a rectangle $\mathcal{U}$ that encloses $\mathcal{T}$. One vertex of $\mathcal{T}$ must coincide with a vertex of $\mathcal{U}$, because each side of $\mathcal{U}$ passes through a vertex of $\mathcal{T}$. Keeping rectangle $\mathcal{U}$ inside rectangle $\mathcal{R}$, we can slide $\mathcal{U}$ and $\mathcal{T}$ so that this vertex of $\mathcal{T}$ coincides with a vertex of $\mathcal{R}$. It is therefore no loss of generality to assume that $\mathcal{T}$ has a vertex at $A$.

If an equilateral triangle $AXY$ exists that has $X$ on $\overline{BC}$ and $Y$ on $\overline{CD}$, then $AXY$ has maximal area among all equilateral triangles that lie inside $\mathcal{R}$. The reason is that any other equilateral triangle with a vertex at $A$ either must have a vertex that lies in or on triangle $ADY$ or must have a vertex that lies in or on triangle $ABX$; this implies a smaller area.

To see that such a triangle exists, and to find its area, let $X$ and $Y$ be points on $\overline{BC}$ and $\overline{CD}$, respectively, such that $\angle XAY$ is a 60-degree angle. Let $\theta = \angle BAX$, so that $AX = a\sec\theta$ and $AY = b\sec(30^\circ - \theta)$. Triangle $AXY$ is equilateral if and only if $AX = AY$, which is equivalent to $a\cos(30^\circ - \theta) = b\cos\theta$, which, according to the subtraction law for cosines, is equivalent to $a\sqrt{3}\cos\theta + a\sin\theta = 2b\cos\theta$. That is, $\tan\theta = \dfrac{2b}{a} - \sqrt{3}$, which lies between $\tan 0^\circ$ and $\tan 30^\circ = \dfrac{1}{\sqrt{3}}$ if and only if $\dfrac{a}{b}$ lies between $\dfrac{\sqrt{3}}{2}$ and $\dfrac{2}{\sqrt{3}}$. Because $\{a, b\} = \{10, 11\}$, equilateral triangle $AXY$ exists. Furthermore, the area of triangle $AXY$ is

$$\frac{\sqrt{3}}{4}AX^2 = \frac{\sqrt{3}}{4}a^2\sec^2\theta = \frac{\sqrt{3}}{4}a^2\left(1+\tan^2\theta\right) = \frac{\sqrt{3}}{4}a^2\left[1+\left(\frac{2b}{a}-\sqrt{3}\right)^2\right],$$

which simplifies to $(a^2+b^2)\sqrt{3} - 3ab = 221\sqrt{3} - 330$. Hence $p + q + r = 554$.