AMERICAN MATHEMATICS COMPETITIONS

AIME SOLUTIONS PAMPHLET
FOR STUDENTS AND TEACHERS

16th ANNUAL
AMERICAN INVITATIONAL MATHEMATICS EXAMINATION
(AIME)

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This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs. geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution. *Remember that reproduction of these solutions is prohibited by copyright.*

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1. (Answer: 025)
The prime factorizations of the given integers are

$$6^6 = 2^6 3^6, \quad 8^8 = 2^{24}, \quad \text{and} \quad 12^{12} = 2^{24}3^{12}.$$

Because 2 and 3 are the only prime factors of the least common multiple of the three numbers, $k = 2^m 3^n$ for some nonnegative integers m and n. In order for the least common multiple of $2^6 3^6$, 2^{24}, and $2^m 3^n$ to be $2^{24}3^{12}$, n must be 12, and m can be any integer from 0 to 24, inclusive. Thus there are 25 acceptable values of k.

2. (Answer: 480)
Points that have integer coordinates are called *lattice points*. The lattice points in question lie within the square defined by $1 \le x \le 30$ and $1 \le y \le 30$. The only lattice points that are *not* included are those for which $2y < x$ or $2x < y$. For positive x and y, these conditions are mutually exclusive. Within the square, the inequality $2y < x$ cannot hold for $y \ge 15$. For each integer y between 1 and 14, inclusive, there are $30 - 2y$ such points that satisfy $2y < x$, namely those for which $2y + 1 \le x \le 30$. The number of points that satisfy $2x < y$ is the same as the number of points that satisfy $2y < x$. The total number of omitted points is therefore

$$2(2 + 4 + \cdots + 28) = 4(1 + 2 + 3 + \cdots + 14) = 420,$$

making the answer $900 - 420 = 480$.

OR

The conditions in the problem can be expressed as $1 \le x \le 30$, $y/2 \le x \le 2y$, and $1 \le y \le 30$. For each value of y from 1 to 15, x must be between $\lceil \frac{y}{2} \rceil$ and $2y$, inclusive, so there are $2y - \lceil \frac{y}{2} \rceil + 1$ values of x. (The value $\lceil r \rceil$ of the *ceiling function* is the smallest integer that is not less than r.) For each value of y from 16 to 30, similar reasoning shows that there are $30 - \lceil \frac{y}{2} \rceil + 1$ values of x. The number of ordered pairs is thus

$$\sum_{y=1}^{15} \left(2y - \left\lceil \frac{y}{2} \right\rceil + 1\right) + \sum_{y=16}^{30} \left(30 - \left\lceil \frac{y}{2} \right\rceil + 1\right)$$

$$= \left(\sum_{y=1}^{15} 2y\right) + 15 + 15 \cdot 31 - \sum_{y=1}^{30} \left\lceil \frac{y}{2} \right\rceil$$

$$= 2\sum_{y=1}^{15} y + 480 - 2\sum_{y=1}^{15} y$$

$$= 480.$$

3. (Answer: 800)

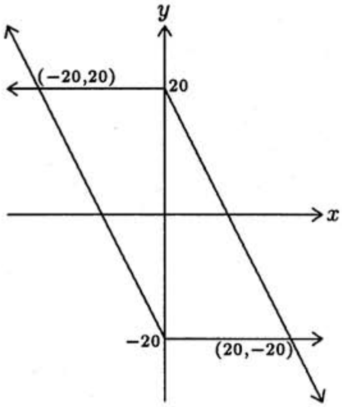
For nonnegative values of x, the equation can be written $y^2 - 400 + 2x(y+20) = 0$, or

$$(y - 20 + 2x)(y + 20) = 0.$$

For nonnegative x, the graph thus consists of two rays. For nonpositive values of x, the equation can be written $y^2 - 400 + 2x(y - 20) = 0$, or

$$(y + 20 + 2x)(y - 20) = 0.$$

This gives two more rays. The bounded region is therefore enclosed by a parallelogram whose height is 40, whose base is 20, and whose area is 800.



Note: Because $(-x, -y)$ satisfies the given equation if and only if (x, y) does, the graph is symmetric with respect to the origin. The requested area is thus twice the area of the triangle enclosed by the y-axis and the two rays found for nonnegative x.

4. (Answer: 017)

Each player must select an odd number of odd-numbered tiles. Because there are five odd-numbered tiles available, one player must select three of them, and the other two players must select one each. The probability that the first player selects three odd-numbered tiles is $10/84$, for there are $\binom{9}{3} = 84$ ways to select three tiles from the nine available, and there are $\binom{5}{3} = 10$ ways to select three odd-numbered tiles from the five available. Given that this event has occurred, the probability that the second player will choose exactly one odd-numbered tile is $12/20$, for there are $\binom{6}{3} = 20$ ways to select three tiles from the six that remain, and there are $\binom{2}{1}\binom{4}{2} = 12$ ways to select one odd-numbered and two even-numbered tiles. Given that the first two players have each selected an odd number of odd-numbered tiles, the third is sure to do the same. Because any player can be the one who selects three odd-numbered tiles, the desired probability is $3(10/84)(12/20) = 3/14$, so $m + n = 17$.

5. (Answer: 040)

Notice that $\dfrac{k(k-1)}{2}$ is even when $k = 4m$ or $k = 4m+1$, and odd otherwise. It follows that

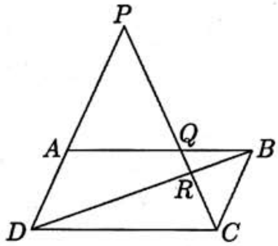
$$A_{4m-1} + A_{4m} = -\frac{(4m-1)(4m-2)}{2} + \frac{4m(4m-1)}{2} = 4m - 1$$

and

$$A_{4m+1} + A_{4m+2} = \frac{(4m+1)4m}{2} - \frac{(4m+2)(4m+1)}{2} = -4m - 1,$$

hence $A_{4m-1} + A_{4m} + A_{4m+1} + A_{4m+2} = -2$. Thus $A_{19} + A_{20} + A_{21} + \cdots + A_{98}$, which is a sum of eighty terms, equals $20(-2) = -40$.

6. (Answer: 308)
The similarity of triangles RBC and RDP implies that $\frac{RC}{RP} = \frac{RB}{RD}$, and the similarity of triangles RBQ and RDC implies that $\frac{RB}{RD} = \frac{RQ}{RC}$. Thus $\frac{RC}{RP} = \frac{RQ}{RC}$, or $RC^2 = RQ \cdot RP = 112 \cdot 847 = 16 \cdot 7 \cdot 7 \cdot 121$. Hence $RC = 4 \cdot 7 \cdot 11 = 308$.



7. (Answer: 196)
Each x_i can be replaced by $2y_i - 1$, where y_i is a positive integer. Because

$$98 = \sum_{i=1}^{4} x_i = \sum_{i=1}^{4}(2y_i - 1) = 2\left(\sum_{i=1}^{4} y_i\right) - 4$$

it follows that $51 = \sum_{i=1}^{4} y_i$. Each such quadruple (y_1, y_2, y_3, y_4) corresponds in a one-to-one fashion to a row of 51 ones that has been separated into four groups by the insertion of three zeros. For example, $(17, 5, 11, 18)$ corresponds to

$$11111111111111111\,0\,11111\,0\,11111111111\,0\,111111111111111111.$$

There are $\binom{50}{3} = 19600$ ways to insert three zeros into the fifty spaces between adjacent ones. Thus there are $n = 19600$ of the requested sums, and $\frac{n}{100} = 196$.

8. (Answer: 618)
Let $m = 1000$. The given sequence is

$$m,\ x,\ m-x,\ 2x-m,\ 2m-3x,\ 5x-3m,\ 5m-8x,\ \ldots .$$

Except for alternating signs, the coefficients of m and x in this sequence appear to belong to the Fibonacci-type sequence 1, 0, 1, 1, 2, 3, 5, 8, 13, 21, ..., in which each term is the sum of its two predecessors. Because the goal is to avoid negative terms in the given sequence, an optimal x satisfies as many of the following inequalities as possible before failing:

$$x < m\ ,\ \tfrac{1}{2}m < x\ ,\ x < \tfrac{2}{3}m\ ,\ \tfrac{3}{5}m < x\ ,\ x < \tfrac{5}{8}m\ ,\ \ldots .$$

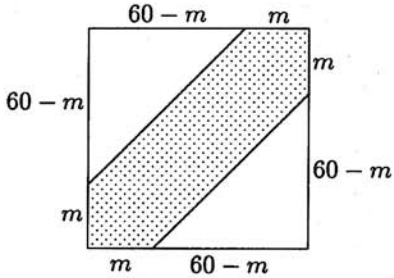
Each inequality involves a ratio of two successive terms of the Fibonacci sequence. Beginning with the fourth inequality, an optimal x must satisfy

$$600 < x\ ,\ x < 625\ ,\ 615 < x\ ,\ x < 620\ ,\ 617 < x\ ,\ \text{and } x < 619.$$

It follows that $x = 618$, which produces the fourteen-term sequence 1000, 618, 382, 236, 146, 90, 56, 34, 22, 12, 10, 2, 8, -6.

Challenge: Prove that the coefficients of m and x do appear unsigned in the Fibonacci sequence.

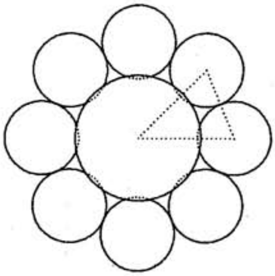
9. (Answer: 087)
In the figure below, points in the square correspond to ordered pairs (x, y) of arrival times, with $0 \leq x \leq 60$ and $0 \leq y \leq 60$. The shaded points correspond to meetings, which occur if and only if $|x-y| \leq m$. The probability of no meeting is $3/5$, which is the ratio of the unshaded region to the area of the whole square. Thus $(60-m)^2 = \frac{3}{5} \cdot 60^2$, whose solutions are $m = 60 \pm 12\sqrt{15}$. Because m must be smaller than 60, it follows that $m = 60 - 12\sqrt{15}$ (about 13.5 minutes), and $a + b + c = 60 + 12 + 15 = 87$.

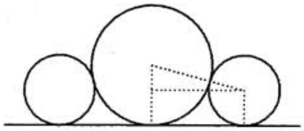


10. (Answer: 152)
Let d be the distance between the center of one of the eight congruent spheres and the center of the regular octagon. Apply the Law of Cosines to the isosceles triangle formed by the center of the octagon and the centers of two congruent tangent spheres (shown below in the top view). This yields $200^2 = d^2 + d^2 - 2d^2 \cos 45° = d^2(2 - \sqrt{2})$, from which follows

$$d^2 = \frac{40000}{2 - \sqrt{2}} = 40000 + 20000\sqrt{2}.$$

Let r be the radius of the ninth sphere. The center of one of the eight congruent spheres, the center of the octagon, and the center of the ninth sphere form a right triangle (shown below in the side view). Apply the Pythagorean Theorem to obtain $d^2 + (r - 100)^2 = (r + 100)^2$, which is equivalent to $d^2 = 400r$. It follows that $r = 100 + 50\sqrt{2}$, and thus $a + b + c = 152$.

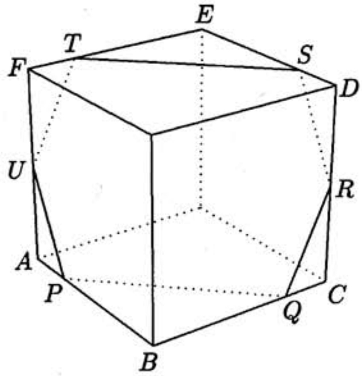

top view


side view

11. (Answer: 525)

Because $BP = BQ$, $\overline{PQ}$ is parallel to $\overline{AC}$. Thus the line through R that is parallel to PQ will intersect $\overline{AF}$ at U so that $AU = CR$. Because the line through R that is parallel to $\overline{PQ}$ is in plane PQR, U is a vertex of the intersection polygon. The midpoint of $\overline{RU}$ is also the center of the cube, so the intersection polygon has point symmetry with respect to the center of the cube. Hence its area is twice the area of isosceles trapezoid $PQRU$, whose altitude is

$$\sqrt{UP^2 - \left(\frac{UR - PQ}{2}\right)^2}.$$



Given the lengths $UR = 20\sqrt{2}$, $PQ = 15\sqrt{2}$, and $UP = \sqrt{125}$, the desired area is found to be

$$(UR + PQ)\sqrt{UP^2 - \left(\frac{UR - PQ}{2}\right)^2} = 35\sqrt{2}\sqrt{\frac{225}{2}} = 525.$$

OR

Set up a coordinate system so that $A = (0,0,0)$, $B = (20,0,0)$, $C = (20,20,0)$, and $D = (20,20,20)$. It follows that $P = (5,0,0)$, $Q = (20,15,0)$, and $R = (20,20,10)$. Plane PQR can be described by an equation $ax + by + cz = d$. Substitute the coordinates of P, Q, and R into this equation to find that

$$d = 5a = 20a + 15b = 20a + 20b + 10c,$$

hence that $a = -b = 2c$. Thus plane PQR is described by $2x - 2y + z = 10$. To find coordinates for the other points where the plane intersects the edges of the cube, replace two of the unknowns by 0 or 20, and solve for the third, which must also be between 0 and 20. This yields the three additional points $S = (15,20,20)$, $T = (0,5,20)$, and $U = (0,0,10)$. The area of hexagon $PQRSTU$ may now be found as above.

12. (Answer: 083)

To see first that there is at most one set of points with the given property, suppose that P', Q', and R' also have the given property. Notice that P' is on $\overline{PE}$ if and only if Q' is on $\overline{QE}$, which is true if and only if R' is on $\overline{RD}$, which is true if and only if P' is on $\overline{PD}$. Conclude that $P' = P$. It follows that $Q' = Q$, because P determines the position of Q on $\overline{EF}$, and $R' = R$, because Q determines the position of R on $\overline{DF}$. Without loss of generality, let $DC = DE = 1$ and $DP = x$. Triangles ABC and DEF have 120-degree rotational symmetry, hence triangle PQR must also. (If this were not true, then a 120-degree rotation would produce another set of points with the given property.) It follows that $EQ = x$ and $PE = 1 - x$. The similarity of triangles DCP and EQP implies that $\dfrac{DP}{EP} = \dfrac{DC}{EQ}$, or $\dfrac{x}{1-x} = \dfrac{1}{x}$. Thus $x^2 = 1 - x$, whose positive solution is $x = \dfrac{1}{2}(\sqrt{5} - 1)$. Apply the Law of Cosines to triangle PQE to obtain

$$\begin{aligned} PQ^2 &= PE^2 + EQ^2 - 2 \cdot PE \cdot EQ \cdot \cos 60^\circ \\ &= (1-x)^2 + x^2 - x(1-x) \\ &= 3x^2 - 3x + 1 \\ &= 3(1-x) - 3x + 1 \\ &= 4 - 6x \\ &= 7 - 3\sqrt{5}. \end{aligned}$$

Therefore

$$\frac{\text{Area}(\triangle ABC)}{\text{Area}(\triangle PQR)} = \frac{AB^2}{PQ^2} = \frac{4}{7 - 3\sqrt{5}} = 7 + 3\sqrt{5},$$

and $a^2 + b^2 + c^2 = 49 + 9 + 25 = 83$.

13. (Answer: 368)

It suffices to express S_{n+1} in terms of S_n and n. The subsets of $\{1,2,3,\ldots,n+1\}$ that do not contain $n+1$ are just the subsets of $\{1,2,3,\ldots,n\}$, and thus the sum of their complex power sums is just S_n. The subsets of $\{1,2,3,\ldots,n+1\}$ that do contain $n+1$ are the subsets of $\{1,2,3,\ldots,n\}$ with $n+1$ adjoined as the greatest element. Notice that $n+1$ will be the only member in one of these subsets, the second member in n of these subsets, and in general will be the k^{th} member in $\binom{n}{k-1}$ of these subsets. The sum of the complex power sums of all subsets of $\{1,2,3,\ldots,n+1\}$ that contain $n+1$ is therefore

$$\begin{aligned} S_n &+ \binom{n}{0}(n+1)i + \binom{n}{1}(n+1)i^2 + \binom{n}{2}(n+1)i^3 + \cdots + \binom{n}{n}(n+1)i^{n+1} \\ &= S_n + i(n+1)\sum_{k=0}^{n}\binom{n}{k}i^k, \end{aligned}$$

which by the Binomial Theorem is equal to $S_n + i(n+1)(1+i)^n$. The desired recursion is therefore $S_{n+1} = 2S_n + i(n+1)(1+i)^n$. Thus

$$S_9 = 2S_8 + 9i(1+i)^8 = -352 - 128i + 9i\cdot 16 = -352 + 16i,$$

so $|p| + |q| = 368$.

Note: To solve the recursion, sum the equations

$$\begin{aligned} S_n &= 2S_{n-1} + in(1+i)^{n-1}, \\ 2S_{n-1} &= 4S_{n-2} + 2i(n-1)(1+i)^{n-2}, \\ 4S_{n-2} &= 8S_{n-3} + 4i(n-2)(1+i)^{n-3}, \\ &\vdots \\ 2^{n-2}S_2 &= 2^{n-1}S_1 + 2^{n-2}\,i\,2(i+1), \text{ and} \\ 2^{n-1}S_1 &= 2^{n-1}\,i \end{aligned}$$

to obtain

$$\begin{aligned} S_n &= i\left(2^{n-1} + 2^{n-2}2(i+1) + \cdots + n(1+i)^{n-1}\right) \\ &= i\,2^{n-1}\left(1 + 2z + 3z^2 + \cdots + n\,z^{n-1}\right), \end{aligned}$$

where $z = \frac{1}{2}(1+i)$. Multiply both sides by $1-z$ to obtain

$$\begin{aligned} (1-z)S_n &= i\,2^{n-1}\left(1 + z + z^2 + \cdots + z^{n-1} - n\,z^n\right) \\ &= i\,2^{n-1}\left[\frac{1-z^n}{1-z} - nz^n\right]. \end{aligned}$$

Then multiply both sides by $(1-z)^{-1} = 1+i$ to obtain

$$S_n = i\,2^{n-1}\left[2i\left(1-\left(\frac{1+i}{2}\right)^n\right) - n(1+i)\left(\frac{1+i}{2}\right)^n\right],$$

which simplifies to $S_n = (n+1+i)(1+i)^{n-1} - 2^n$.

Here is a combinatorial approach to evaluating S_n: Observe that, for $1 \le k \le n$ and $1 \le r \le k$, the term ki^r occurs in the sum of complex power sums for each choice of $1 \le a_1 < a_2 \cdots \le n$ in which $a_r = k$. Thus $S_n = \sum_{k=1}^{n}\sum_{r=1}^{k} W(k,r)ki^r$, where $W(k,r)$ is the number of subsets $\{a_1, a_2, \ldots\}$ of $\{1, 2, \ldots, n\}$ in which the r^{th} smallest element is k. Notice that $W(k,r) = \binom{k-1}{r-1}2^{n-k}$, because there are $\binom{k-1}{r-1}$ ways to choose $1 \le a_1 < a_2 < \cdots < a_{r-1} \le k-1$ and then 2^{n-k} ways to choose $\{a_{r+1}, a_{r+2}, \ldots\} \subseteq \{k+1, \ldots, n\}$. Hence, by the Binomial Theorem,

$$\begin{aligned}
S_n &= \sum_{k=1}^{n}\sum_{r=1}^{k}\binom{k-1}{r-1}2^{n-k}k\,i^r \\
&= i\,2^{n-1}\sum_{k=1}^{n}k2^{-(k-1)}\sum_{r=1}^{k}\binom{k-1}{r-1}i^{r-1} \\
&= i\,2^{n-1}\sum_{k=1}^{n}kz^{k-1},
\end{aligned}$$

where $z = \frac{1}{2}(1+i)$. Continue as above.

14. (Answer: 130)

First notice that $2mnp = (m+2)(n+2)(p+2)$ implies $\dfrac{2m}{m+2} = \dfrac{(n+2)(p+2)}{np} > 1$, which shows that $m \ge 3$. Next rewrite the equation as

$$\begin{aligned}
&2mnp = mnp + 2(mn+np+mp) + 4(m+n+p) + 8, \\
&mnp - 2(mn+np+mp) + 4(m+n+p) - 8 = 8(m+n+p), \\
&(m-2)(n-2)(p-2) = 8(m+n+p),
\end{aligned}$$

which suggests replacing $m-2$, $n-2$, and $p-2$ by the positive integers a, b, and c, respectively. Notice that $1 \le a$. The problem is now to find the largest c that satisfies the equation $abc = 8(a+b+c+6)$, which can be rewritten $\dfrac{c}{8} = \dfrac{a+b+6}{ab-8}$. Because $(a-1)(b-1)$ is nonnegative, it follows that $a+b \le ab+1$, hence that

$$\frac{c}{8} = \frac{a+b+6}{ab-8} \le \frac{ab+1+6}{ab-8} = \frac{ab-8+15}{ab-8} = 1 + \frac{15}{ab-8}\,.$$

This shows that c can be no larger than $8 \cdot 16 = 128$, and that c attains this value if $ab = 9$ and $a+b = ab+1 = 10$. Thus $c = 128$ when $a = 1$ and $b = 9$, and $m = 3$, $n = 11$, and $p = 130$ are the dimensions of a possible box.

15. (Answer: 761)

Let $A_n = \{1, 2, 3, \ldots, n\}$ and D_n be the set of dominos that can be formed using integers in A_n. Each k in A_n appears in $2(n-1)$ dominos in D_n, hence appears at most $n-1$ times in a proper sequence from D_n. Except possibly for the integers i and j that begin and end a proper sequence, every integer appears an even number of times in the sequence. Thus, if n is even, each integer different from i and j appears on at most $n-2$ dominos in the sequence, because $n-2$ is even, and i and j themselves appear on at most $n-1$ dominos each. This gives an upper bound of

$$\frac{1}{2}\left[(n-2)^2 + 2(n-1)\right] = \frac{n^2 - 2n + 2}{2}$$

dominos in the longest proper sequence in D_n. This bound is in fact attained for every even n. It is easy to verify this for $n = 2$, so assume inductively that a sequence of this length has been found for a particular value of n. Without loss of generality, assume $i = 1$ and $j = 2$, and let ${}_pX_{p+2}$ denote a four-domino sequence of the form $(p, n+1)(n+1, p+1)(p+1, n+2)(n+2, p+2)$. By appending

$${}_2X_4\ ,\ {}_4X_6\ ,\ \ldots\ ,\ {}_{n-2}X_n\ ,\ (n, n+1)(n+1, 1)(1, n+2)(n+2, 2)$$

to the given proper sequence, a proper sequence of length

$$\frac{n^2 - 2n + 2}{2} + 4 \cdot \frac{n-2}{2} + 4 = \frac{n^2 + 2n + 2}{2} = \frac{(n+2)^2 - 2(n+2) + 2}{2}$$

is obtained that starts at $i = 1$ and ends at $j = 2$. This completes the inductive proof. In particular, the longest proper sequence when $n = 40$ is 761.

OR

A proper sequence can be represented by writing the common coordinates of adjacent ordered pairs once. For example, represent (4,7), (7,3), (3,5) as 4,7,3,5. Label the vertices of a regular n-gon 1, 2, 3, ..., n. Each domino is thereby represented by a directed segment from one vertex of the n-gon to another, and a proper sequence is represented as a path that retraces no segment. Each time that such a path reaches a non-terminal vertex, it must leave it. Thus, when n is even, it is not possible for such a path to trace every segment, for an odd number of segments emanate from each vertex. By removing $\frac{1}{2}(n-2)$ suitable segments, however, it can be arranged that $n-2$ segments will emanate from $n-2$ of the vertices, and that an odd number of segments will emanate from exactly two of the vertices. In this situation, a path can be found that traces every remaining segment exactly once, starting at one of the two exceptional vertices and finishing at the other. This path will have length $\binom{n}{2} - \frac{1}{2}(n-2)$, which is 761 when $n = 40$.

Note: When n is odd, a proper sequence of length $\binom{n}{2}$ can be found using the dominos of D_n. In this case, the second coordinate of the final domino equals the first coordinate of the first domino. In the language of graph theory, this is an example of an *Eulerian circuit*.