AIME SOLUTIONS PAMPHLET

FOR STUDENTS AND TEACHERS

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1984

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This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs. geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

It is hoped that teachers will find the time to share these solutions with their students, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution.

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1. (Answer: 93)

The sum of an arithmetic progression is the product of the number of terms and the arithmetic average of the first and last terms. Therefore, $a_1 + a_2 + \cdots + a_{98} = 98(a_1 + a_{98})/2 = 49(a_1 + a_{98}) = 137$. Similarly, $a_2 + a_4 + \cdots + a_{98} = 49(a_2 + a_{98})/2 = 49(a_1 + 1 + a_{98})/2 = (49(a_1 + a_{98})/2) + (49/2) = (137/2) + (49/2) = 93$.

Alternate Solution. Separating the terms of odd and even subscripts, let $S_o = a_1 + a_3 + \cdots + a_{97}$ and $S_e = a_2 + a_4 + \cdots + a_{98}$. Then each of these series has 49 terms, and since $a_{n+1} = a_n + 1$ for all $n \geq 1$, we have $S_o = S_e - 49$. Furthermore, $S_e + S_o = 137$. Substituting for S_o in the last equation, and solving for S_e, one finds that $S_e = 93$.

2. (592)

Note that n is a common multiple of 5 and 3. As a multiple of 5, n must end in 0, as the digit 5 is not allowed. As a multiple of 3, n must contain a number of 8's equal to a multiple of 3. Hence, in view of the minimality requirement, n = 8880, and 8880/15 = 592 is the answer to the problem.

3. (144)

Let T denote the area of ΔABC, and denote by T_1, T_2 and T_3 the areas of t_1, t_2 and t_3, respectively. Moreover, let c be the length of AB, and let c_1, c_2 and c_3 be the lengths of the bases parallel to AB of t_1, t_2 and t_3, respectively. Then, in view of the similarity of the four triangles, one has

$$\frac{\sqrt{T_1}}{\sqrt{T}} = \frac{c_1}{c}, \quad \frac{\sqrt{T_2}}{\sqrt{T}} = \frac{c_2}{c} \quad \text{and} \quad \frac{\sqrt{T_3}}{\sqrt{T}} = \frac{c_3}{c}.$$

Moreover, since $c_1 + c_2 + c_3 = c$, it follows that

$$\frac{\sqrt{T_1}}{\sqrt{T}} + \frac{\sqrt{T_2}}{\sqrt{T}} + \frac{\sqrt{T_3}}{\sqrt{T}} = \frac{c_1 + c_2 + c_3}{c} = 1,$$

and therefore, $T = (\sqrt{T_1} + \sqrt{T_2} + \sqrt{T_3})^2 = (\sqrt{4} + \sqrt{9} + \sqrt{49})^2 = 144$.

Query: Can you extend the above result to higher dimensions?

4. (649)

Let n denote the number of positive integers in S, and let m denote their sum. Then, on the basis of the information given,

$$\frac{m}{n} = 56 \qquad \text{and} \qquad \frac{m - 68}{n - 1} = 55 .$$

Solving these equations simultaneously yields $n = 13$ and $m = 728$. Now, to maximize the largest element of S, one must minimize the others. To attain this, S must contain eleven 1's, a 68 and a 649, since $728 - 11 - 68 = 649$.

5. (512)

Adding the two equations and using standard log properties yields

$$\log_8 a + \log_8 b + \log_4 a^2 + \log_4 b^2 = \log_8(ab) + 2\log_4(ab) = 12.$$

Moreover, since $\log_8 x = (\log_2 x)/(\log_2 8) = \frac{1}{3}\log_2 x$, and similarly, $\log_4 x = (\log_2 x)/(\log_2 4) = \frac{1}{2}\log_2 x$, the above equation is equivalent to

$$\frac{4}{3}\log_2(ab) = 12.$$

It follows that $\log_2(ab) = 9$, and hence $ab = 2^9 = 512$.

6. (24)

First observe that the three circles are disjoint, i.e. their centers are more than 6 units apart. Next note that any line through (17,76) will divide the area of the circle centered there evenly. Thus the problem reduces to finding the value of m for which the line given by

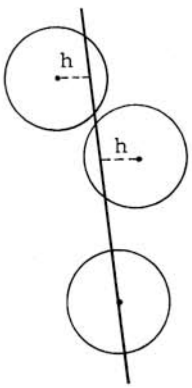
(1) $$y - 76 = m(x - 17)$$

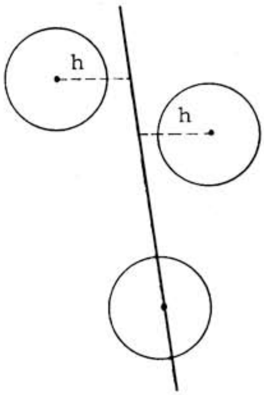
also divides the areas of the other two circles in the desired manner. To this end, note that such a line must pass exactly as far to the right of (14,92) as it passes to the left of (19,84). Denoting this distance by h, it follows that $(14+h, 92)$ and $(19-h, 84)$ must satisfy Equation (1), i.e.

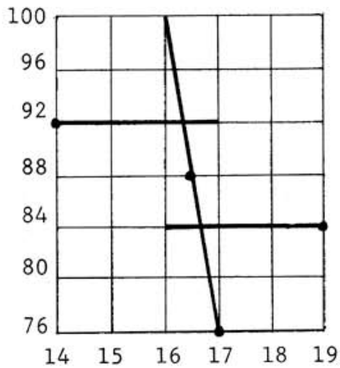
(2) $$\frac{16}{h-3} = m = \frac{8}{2-h},$$

from which $h = \frac{7}{3}$, $m = -24$ and the answer to the problem is 24.

Note. As shown in the first figure below, in this case the uniqueness of the solution is assured by the fact that $h < 3$, the common length of the circles' radii. If the circles were positioned differently with respect to one another, or if the value of h were larger than 3, as exemplified in the second figure below, then the resulting line would not necessarily yield a unique solution to the problem.







Alternate Solution. Note that the lines passing through (16.5 , 88), the center of symmetry of the two circles centered at (14,92) and (19,84), divide the areas of these two circles in the manner desired. Consequently, the line through (16.5 , 88) and (17,76), whose slope is $(88 - 76)/(16.5 - 17) = -24$, provides the answer to the problem. The uniqueness of the solution follows from observing that it intersects the radius from (14,92) to (17,92) of one circle and the radius from (16,84) to (19,84) of the other circle, as shown in the third figure above. (Note the difference in scales.)

7. (997)

Rather than assaulting $f(84)$ directly, it is advantageous to start with values of $f(n)$ near $n = 1000$, and search for a pattern when n is less than 1000, thereby utilizing the recursive definition of f.

Indeed, one finds that

$$
(1)\qquad
\begin{aligned}
f(999) &= f(f(1004)) = f(1001) = 998\,,\\
f(998) &= f(f(1003)) = f(1000) = 997\,,\\
f(997) &= f(f(1002)) = f(999) \ = 998\,,\\
f(996) &= f(f(1001)) = f(998) \ = 997\,,\\
f(995) &= f(f(1000)) = f(997) \ = 998\,,
\end{aligned}
$$

on the basis of which one may conjecture that

$$
(2)\qquad f(n) = \begin{cases} 997\,, & \text{if } n \text{ is even and } n < 1000\,,\\ 998\,, & \text{if } n \text{ is odd and } n < 1000\,. \end{cases}
$$

To prove (2), it is convenient to use downward induction. In this, the inductive step is to prove (2) for n, assuming that it is true for all m, $n < m < 1000$. Since the definition relates $f(n)$ to $f(n+5)$, we can do the inductive step only when $n+5 < 1000$, that is, we must verify the truth of (2) for $n = 999, 998, \ldots, 995$ separately. This was done in (1). Now for $n < 995$,

$$
f(n) = f(f(n+5)) = \begin{cases} f(997) = 998\,, & \text{if } n+5 \text{ is even}\,,\\ f(998) = 997\,, & \text{if } n+5 \text{ is odd}\,. \end{cases}
$$

Noting that n is even when $n+5$ is odd, and that n is odd when $n+5$ is even, completes the proof. In particular, it follows from (2) that $f(84) = 997$.

8. (160)

Let $w = z^3$. Then the given equation reduces to

$$w^2 + w + 1 = 0\,,$$

whose solutions are $(-1+i\sqrt{3})/2$ and $(-1-i\sqrt{3})/2$, with arguments of $120°$ and $240°$, respectively. From these, one finds the following six values for the argument of z:

$$\frac{120°}{3},\ \frac{120°+360°}{3},\ \frac{120°+720°}{3},\ \frac{240°}{3},\ \frac{240°+360°}{3},\ \frac{240°+720°}{3}.$$

Clearly, only the second one of these, $160°$, is between $90°$ and $180°$.

Alternate Solution. Multiplying the given equation by $z^3 - 1 = 0$ yields

$$z^9 - 1 = 0 ,$$

whose solutions are the ninth roots of unity :

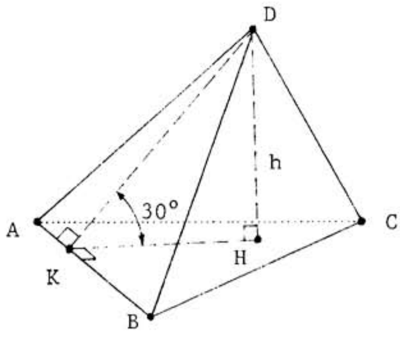
$$z_n = \cos(n\cdot 40^\circ) + i\sin(n\cdot 40^\circ) , \quad n = 0, 1, 2, \ldots, 8 .$$

Of these, only z_3 and z_4 are in the second quadrant. However, since the solutions of $z^9 - 1 = 0$ are distinct, and since z_3 is a solution of $z^3 - 1 = 0$, it cannot be a solution of the original equation. It follows that the desired root is z_4, with degree measure 160 .

9. (20)

Let V be the volume of tetrahedron $ABCD$ and h the altitude of the tetrahedron corresponding to D. Then $V = h\cdot \text{area}(\Delta ABC) / 3$, which will be determined upon finding h.

By definition, for the angle between faces ABC and ABD to be 30°, planes perpendicular to AB must cut the two faces in rays which form a 30° angle. Choose such a plane through D, let K be its intersection with AB, and let H be the point on the line of intersection of the plane chosen and the plane of ΔABC so that $DH \perp KH$. These are shown in the adjoining figure.



Since $DK \perp AB$, we find that $DK = 2\cdot \text{area}(\Delta ABD) / AB = 8\text{cm}$. Moreover, since ΔDKH is a 30°-60°-90° triangle, it follows that $h = DH = DK/2 = 4\text{cm}$. Consequently, substituting into the formula given in the first paragraph, we find that $V = 20\text{cm}^3$.

10. (119)

Given that $s = 30 + 4c - w > 80$, the problem calls for finding the lowest value of s, for which the corresponding value of c is unique. To this end, first observe that if $c + w \le 25$, then by increasing the number of correct answers by 1 and the number of wrong answers by 4, one attains the same score. (This is made possible by the equivalence of the above inequality to $(c+1) + (w+4) \le 30$). Consequently, one must have

(1) $$c + w \geq 26 .$$

Next observe that

(2) $$w \leq 3 ,$$

for otherwise one could reduce the number of wrong answers by 4 and the number of correct answers by 1, and still attain the same score. (The latter is possible since $s > 80$ clearly implies that $c \geq 13$.) Now to minimize s, we must minimize c and maximize w, subject to Inequalities (1) and (2) above. This leads to $w = 3$, $c = 23$ and $s = 30 + 4 \cdot 23 - 3 = 119$.

11. (106)

The 12 trees can be planted in $12!$ orders. Let k be the number of orders in which no two birch trees are adjacent to one another. The probability we need is $k/(12!)$. To find k, we will count the number of patterns

$$\underset{1}{\_\_} \; N \; \underset{2}{\_\_} \; N \; \underset{3}{\_\_} \; N \; \underset{4}{\_\_} \; N \; \underset{5}{\_\_} \; N \; \underset{6}{\_\_} \; N \; \underset{7}{\_\_} \; N \; \underset{8}{\_\_}$$

where the 7 N's denote nonbirch (i.e., maple and oak) trees, and slots 1 through 8 are to be occupied by birch trees, at most one in each slot. There are $7!$ orders for the nonbirch trees, and for each ordering of them there are $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4$ ways to place the birch trees. Thus, we find that $k = (7!) \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4$, $\frac{m}{n} = \frac{(7!) \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{12!} = \frac{7}{99}$ and $m + n = 106$.

Note. We have assumed in this solution that each tree is distinguishable. The problem can also be interpreted to mean that trees are distinguishable if and only if they are of different species. In that case, the calculations (i.e., the numerator and the denominator) are different, but the probability turns out to be the same.

12. (401)

First we use the given equations to find various numbers in the domain to which f assigns the same values. We find that

(1) $$f(x) = f(2 + (x-2)) = f(2 - (x-2)) = f(4-x)$$

and that

(2) $$f(4-x) = f(7 - (x+3)) = f(7 + (x+3)) = f(x+10).$$

From Equations (1) and (2) it follows that

(3) $$f(x+10) = f(x).$$

Replacing x by $x+10$ and then by $x-10$ in Equation (3), we get $f(x+10) = f(x+20)$ and $f(x-10) = f(x)$. Continuing in this way, it follows that

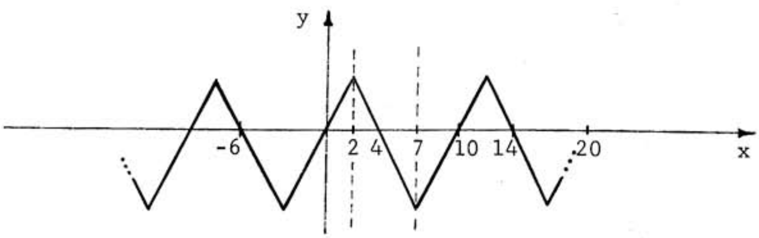
(4) $$f(x+10n) = f(x), \quad \text{for } n = \pm1, \pm2, \pm3, \ldots.$$

Since $f(0) = 0$, Equation (4) implies that

$$f(\pm10) = f(\pm20) = \cdots = f(\pm1000) = 0,$$

necessitating a total of 201 roots for the equation $f(x) = 0$ in the closed interval $[-1000, 1000]$.

Next note that by setting $x=0$ in Equation (1), $f(4) = f(0) = 0$ follows. Therefore, setting $x = 4$ in Equation (4), we obtain 200 more roots for $f(x) = 0$ at $x = -996, -986, \ldots, -6, 4, 14, \ldots, 994$. Since the zig-zag function pictured below satisfies the given conditions and has precisely these and no other roots, the answer to the problem is 401.



Note. Geometrically, the conditions imply symmetry with respect to the lines $x = 2$ and $x = 7$. The solution above shows how to draw consequences from these symmetries through algebra. More generally, one can prove that if a function is symmetric with respect to $x = a$ and $x = b > a$, then it must also have translational symmetry for every integral multiple of $2(b-a)$. This fact can also be proven geometrically, leading to an alternate formulation of the solution.

13. (15)

In order to simplify the notation, let

(1) $a = \cot^{-1} 3$, $b = \cot^{-1} 7$, $c = \cot^{-1} 13$, $d = \cot^{-1} 21$,

(2) $e = a + b$ and $f = c + d$.

Then, in view of (1) and (2) above, the problem is equivalent to finding the value of $10\cot(e+f)$.

It is easy to show that in general,

(3) $$\cot(x+y) = \frac{(\cot x)(\cot y) - 1}{\cot x + \cot y}.$$

Moreover, even if $\cot^{-1}$ is viewed as a multiple valued function,

(4) $$\cot(\cot^{-1} x) = x, \quad \text{for all real } x.$$

Utilizing Equations (1) through (4) above, we find that

$$\cot e = \cot(a+b) = \frac{3\cdot 7 - 1}{3 + 7} = 2, \quad \cot f = \cot(c+d) = \frac{13\cdot 21 - 1}{13 + 21} = 8,$$

$$\cot(e+f) = \frac{2\cdot 8 - 1}{2 + 8} = \frac{3}{2},$$

and therefore, $10\cot(e+f) = 15$ is the answer to the problem.

Note. More generally, since the numbers 3, 7, 13 and 21 are all of the form $1+n+n^2$, one may attempt to express $\cot^{-1}(1+n+n^2)$ more advantageously. Indeed, it is not difficult to show that

$$\cot^{-1}(1+n+n^2) = \tan^{-1}(n+1) - \tan^{-1} n$$

within an integral multiple of π. Consequently, if k is a positive integer, then

$$\sum_{n=1}^{k} \cot^{-1}(1+n+n^2) = \tan^{-1}(k+1) - \tan^{-1} 1,$$

within an integral multiple of π. From this, one can prove that

$$\cot\left(\sum_{n=1}^{k} \cot^{-1}(1+n+n^2)\right) = \frac{k+2}{k}.$$

14. (38)

We will show that if k is an even integer and if $k \geq 40$, then k is expressible as the sum of two composites. This leaves 38 as the candidate for the largest even integer not expressible in such manner; it is easy to check that 38 indeed satisfies this requirement. The proof of our claim for $k \geq 40$ hinges on the fact that if n is odd and greater than 1, then $5n$ is an odd composite ending in 5. So, to express k as desired, it suffices to find small odd composites ending in 5, 7, 9, 1 and 3, and to add these to numbers of the form $5n$. Indeed, 15, 27, 9, 21 and 33 will satisfy the above condition, and in each of the following cases one can find an odd integer n, $n > 1$, such that

if k ends in 0 (i.e. 40, 50, ...), then $k = 15 + 5n$,
if k ends in 2 (i.e. 42, 52, ...), then $k = 27 + 5n$,
if k ends in 4 (i.e. 44, 54, ...), then $k = 9 + 5n$,
if k ends in 6 (i.e. 46, 56, ...), then $k = 21 + 5n$,
if k ends in 8 (i.e. 48, 58, ...), then $k = 33 + 5n$.

Alternate Solution. First observe that $6n + 9$ is an odd composite number for $n = 0, 1, 2, \ldots$. Now partition the set of even positive integers into three residue classes, modulo 6, and note that in each of the following cases the indicated decompositions are satisfied by some nonnegative integer, n:

if $k \equiv 0 \pmod 6$ and $k \geq 18$, then $k = 9 + (6n+9)$,
if $k \equiv 2 \pmod 6$ and $k \geq 44$, then $k = 35 + (6n+9)$,
if $k \equiv 4 \pmod 6$ and $k \geq 34$, then $k = 25 + (6n+9)$.

This takes care of all even integers greater than 38. Checking again shows that 38 is the answer to the problem.

15. (36)

The claim that the given system of equations is satisfied by x^2, y^2, z^2 and w^2 is equivalent to claiming that

$$(1) \qquad \frac{x^2}{t-1} + \frac{y^2}{t-9} + \frac{z^2}{t-25} + \frac{w^2}{t-49} = 1$$

is satisfied by $t = 4, 16, 36$ and 64. Multiplying to clear fractions, we find that for all values of t for which it is defined (i.e., $t \neq 1, 9, 25$ and 49), Equation (1) is equivalent to the polynomial equation

$$(t-1)(t-9)(t-25)(t-49)$$

$$- x^2(t-9)(t-25)(t-49) - y^2(t-1)(t-25)(t-49) \tag{2}$$

$$- z^2(t-1)(t-9)(t-49) - w^2(t-1)(t-9)(t-25) = 0,$$

where the left member may be viewed as a fourth degree polynomial in t. Since $t = 4, 16, 36$ and 64 are known to be roots, and a 4^{th} degree polynomial can have at most 4 roots, these must be all the roots. It follows that (2) is equivalent to

$$(t-4)(t-16)(t-36)(t-64) = 0. \tag{3}$$

Since the coefficient of t^4 is 1 in both (2) and (3), one may conclude that the coefficients of the other powers of t must also be the same. In particular, equating the coefficients of t^3, we have

$$1 + 9 + 25 + 49 + x^2 + y^2 + z^2 + w^2 = 4 + 16 + 36 + 64,$$

from which $x^2 + y^2 + z^2 + w^2 = 36$.

Note. By equating the left members of Equations (2) and (3) and letting $t = 1$, one also finds that $x^2 = 11025/1024$. Similarly, letting $t = 9$, 25 and 49 in succession yields $y^2 = 10395/1024$, $z^2 = 9009/1024$ and $w^2 = 6435/1024$. One can show that these values indeed satisfy the given system of equations, and that their sum is 36.