2nd ANNUAL AMERICAN INVITATIONAL MATHEMATICS EXAMINATION (AIME)

TUESDAY, MARCH 20, 1984

A Prize Examination Sponsored by:
MATHEMATICAL ASSOCIATION OF AMERICA
SOCIETY OF ACTUARIES
MU ALPHA THETA
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
CASUALTY ACTUARIAL SOCIETY

INSTRUCTIONS

1. *Do not open* this booklet until told to do so.
2. This is a 15 question, 2.5 hour examination with integer answers. Your score will be the number of answers you get right. There is no partial credit.
3. All your answers, and certain other information, are to be recorded on a computer card. Your Examination Manager will instruct you how to fill out the card after you have finished with these instructions. Only the computer card and this cover sheet will be collected from you.
4. Scratch paper, graph paper, ruler, compass and eraser are permitted. *Calculators and slide rules are not permitted.*
5. Please print the following:

____
Last Name   First Name   Middle Initial

____
Home Street Address

____
City   State or Province   Zip or Postcode

______________________   ________   ________
Home Phone including Area Code   Sex (M or F)   Your Age

______________________   ________
Full Name of School   Grade Level (e.g., 11)

6. My score on the 1984 AHSME was ☐

   I took the 1984 AHSME on ______________________ .
   (date)

7. This AIME is the qualifying examination for the U.S.A. Mathematical Olympiad (USAMO) to be given on May 1, 1984. Please check one box:

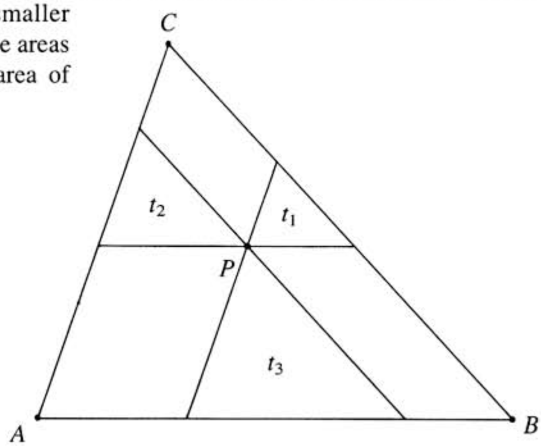
   If I qualify for the USAMO, I agree to take it. YES ☐ NO ☐

   Your school must also agree to administer the USAMO before you can take it.

1. Find the value of $a_2 + a_4 + a_6 + \cdots + a_{98}$ if $a_1, a_2, a_3, \ldots$ is an arithmetic progression with common difference 1, and $a_1 + a_2 + a_3 + \cdots + a_{98} = 137$.

2. The integer n is the smallest positive multiple of 15 such that every digit of n is either 0 or 8. Compute $\frac{n}{15}$.

3. A point P is chosen in the interior of ΔABC so that when lines are drawn through P parallel to the sides of ΔABC, the resulting smaller triangles, t_1, t_2 and t_3 in the figure, have areas 4, 9 and 49, respectively. Find the area of ΔABC.



4. Let S be a list of positive integers—not necessarily distinct—in which the number 68 appears. The average (arithmetic mean) of the numbers in S is 56. However, if 68 is removed, the average of the remaining numbers drops to 55. What is the largest number that can appear in S?

5. Determine the value of ab if $\log_8 a + \log_4 b^2 = 5$ and $\log_8 b + \log_4 a^2 = 7$.

6. Three circles, each of radius 3, are drawn with centers at (14,92), (17,76) and (19,84). A line passing through (17,76) is such that the total area of the parts of the three circles to one side of the line is equal to the total area of the parts of the three circles to the other side of it. What is the absolute value of the slope of this line?

7. The function f is defined on the set of integers and satisfies

$$f(n) = \begin{cases} n-3, & \text{if } n \geq 1000, \\ f(f(n+5)), & \text{if } n < 1000. \end{cases}$$

Find $f(84)$.

8. The equation $z^6 + z^3 + 1 = 0$ has one complex root with argument (angle) θ between $90°$ and $180°$ in the complex plane. Determine the degree measure of θ.

9. In tetrahedron $ABCD$, edge AB has length 3 cm. The area of face ABC is 15 cm^2 and the area of face ABD is 12 cm^2. These two faces meet each other at a $30°$ angle. Find the volume of the tetrahedron in cm^3.

10. Mary told John her score on the American High School Mathematics Examination (AHSME), which was over 80. From this, John was able to determine the number of problems Mary solved correctly. If Mary's score had been any lower, but still over 80, John could not have determined this. What was Mary's score? (Recall that the AHSME consists of 30 multiple-choice problems and that one's score, s, is computed by the formula $s = 30 + 4c - w$, where c is the number of correct and w is the number of wrong answers; students are not penalized for problems left unanswered.)

11. A gardener plants three maple trees, four oak trees and five birch trees in a row. He plants them in random order, each arrangement being equally likely. Let $\frac{m}{n}$ in lowest terms be the probability that no two birch trees are next to one another. Find $m + n$.

12. A function f is defined for all real numbers and satisfies

$$f(2+x) = f(2-x) \qquad \text{and} \qquad f(7+x) = f(7-x)$$

for all real x. If $x = 0$ is a root of $f(x) = 0$, what is the least number of roots $f(x) = 0$ must have in the interval $-1000 \leq x \leq 1000$?

13. Find the value of $10\cot(\cot^{-1}3 + \cot^{-1}7 + \cot^{-1}13 + \cot^{-1}21)$.

14. What is the largest even integer which cannot be written as the sum of two odd composite numbers? (Recall that a positive integer is said to be composite if it is divisible by at least one positive integer other than 1 and itself.)

15. Determine $x^2 + y^2 + z^2 + w^2$ if

$$\frac{x^2}{2^2-1^2} + \frac{y^2}{2^2-3^2} + \frac{z^2}{2^2-5^2} + \frac{w^2}{2^2-7^2} = 1,$$

$$\frac{x^2}{4^2-1^2} + \frac{y^2}{4^2-3^2} + \frac{z^2}{4^2-5^2} + \frac{w^2}{4^2-7^2} = 1,$$

$$\frac{x^2}{6^2-1^2} + \frac{y^2}{6^2-3^2} + \frac{z^2}{6^2-5^2} + \frac{w^2}{6^2-7^2} = 1,$$

$$\frac{x^2}{8^2-1^2} + \frac{y^2}{8^2-3^2} + \frac{z^2}{8^2-5^2} + \frac{w^2}{8^2-7^2} = 1.$$

Students and teachers with questions or comments about this AIME may write to:

Professor George Berzsenyi, AIME Chairman
Department of Mathematics
Lamar University
Beaumont, TX 77710

Questions about administrative arrangements for the AIME should be addressed to:

Professor Walter E. Mientka, Executive Director
MAA Committee on High School Contests
Department of Mathematics and Statistics
University of Nebraska
Lincoln, NE 68588-0322

Information about ordering past copies of other examinations given by the Committee is found on the back cover of this examination.

PUBLICATIONS LIST

The following publications are available for purchase by those interested in supplementary practice examination materials. Prices effective until September 1, 1984.

A. The American High School Mathematics Examination—Prior Examinations, 1972-1984, Spanish editions, 1978-84.
   Specimen sets of prior examinations. Each set contains a question booklet and a solution pamphlet. 40¢ each set; specify year desired.

B. The American Invitational Mathematics Examination—1983-84 AIME and a solution pamphlet. 50¢ each set.

C. The U.S.A. and International Mathematical Olympiads, 1976-1983. These pamphlets contain the problems and solutions to the 1976-83 Olympiads. 50¢ each pamphlet; specify year desired.

D. *Contest Problem Book I* (at $6.50 each) contains AHSME questions and solutions for 1950-1960.
   Contest Problem Book II (at $6.50 each) contains AHSME questions and solutions for 1961-1965.
   Contest Problem Book III (at $7.50 each) contains AHSME questions and solutions for 1966-1972.
   Contest Problem Book IV (at $8.50 each) contains AHSME questions and solutions for 1973-1982.
   International Mathematical Olympiads (at $7.50 each) contains IMO questions and solutions for 1959-1977.

E. The *Arbelos* (at $4.00 per subscription-5 issues per year) is a new journal containing short articles and challenging problems for gifted students. Available issues 1982-83 and 1983-84.

ORDER INFORMATION

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ORDER FROM: Dr. Walter E. Mientka, Executive Director
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