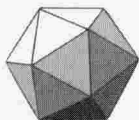


MATHEMATICAL ASSOCIATION OF AMERICA  
AMERICAN MATHEMATICS COMPETITIONS

PRESENTED BY THE AKAMAI FOUNDATION



19th Annual (*Alternate*)

AMERICAN INVITATIONAL  
MATHEMATICS EXAMINATION  
(AIME)

Tuesday, April 10, 2001

1. DO NOT OPEN THIS BOOKLET UNTIL TOLD TO DO SO BY YOUR PROCTOR.
2. This is a 15-question, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers; i.e., there is neither partial credit nor a penalty for wrong answers.
3. No aids other than scratch paper, graph paper, ruler, compass, and protractor are permitted. In particular, calculators are not permitted.
4. A combination of the AIME and the American Mathematics Contest 10 or the American Mathematics Contest 12 scores are used to determine eligibility for participation in the U.S.A. Mathematical Olympiad (USAMO). The USAMO will be given on TUESDAY, May 1, 2001.
5. Record all of your answers, and certain other information, on the AIME answer form. Only the answer form will be collected from you.

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- Let  $N$  be the largest positive integer with the following property: reading from left to right, each pair of consecutive digits of  $N$  forms a perfect square. What are the leftmost three digits of  $N$ ?
- Each of the 2001 students at a high school studies either Spanish or French, and some study both. The number who study Spanish is between 80 percent and 85 percent of the school population, and the number who study French is between 30 percent and 40 percent. Let  $m$  be the smallest number of students who could study both languages, and let  $M$  be the largest number of students who could study both languages. Find  $M - m$ .

- Given that

$$x_1 = 211,$$

$$x_2 = 375,$$

$$x_3 = 420,$$

$$x_4 = 523, \text{ and}$$

$$x_n = x_{n-1} - x_{n-2} + x_{n-3} - x_{n-4} \text{ when } n \geq 5,$$

find the value of  $x_{531} + x_{753} + x_{975}$ .

- Let  $R = (8, 6)$ . The lines whose equations are  $8y = 15x$  and  $10y = 3x$  contain points  $P$  and  $Q$ , respectively, such that  $R$  is the midpoint of  $\overline{PQ}$ . The length  $PQ$  equals  $m/n$ , where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .
- A set of positive numbers has the *triangle property* if it has three distinct elements that are the lengths of the sides of a triangle whose area is positive. Consider sets  $\{4, 5, 6, \dots, n\}$  of consecutive positive integers, all of whose ten-element subsets have the triangle property. What is the largest possible value of  $n$ ?
- Square  $ABCD$  is inscribed in a circle. Square  $EFGH$  has vertices  $E$  and  $F$  on  $\overline{CD}$  and vertices  $G$  and  $H$  on the circle. The ratio of the area of square  $EFGH$  to the area of square  $ABCD$  can be expressed as  $m/n$  where  $m$  and  $n$  are relatively prime positive integers and  $m < n$ . Find  $10n + m$ .
- Let  $\triangle PQR$  be a right triangle with  $PQ = 90$ ,  $PR = 120$ , and  $QR = 150$ . Let  $C_1$  be the inscribed circle. Construct  $\overline{ST}$ , with  $S$  on  $\overline{PR}$  and  $T$  on  $\overline{QR}$ , such that  $\overline{ST}$  is perpendicular to  $\overline{PR}$  and tangent to  $C_1$ . Construct  $\overline{UV}$  with  $U$  on  $\overline{PQ}$  and  $V$  on  $\overline{QR}$  such that  $\overline{UV}$  is perpendicular to  $\overline{PQ}$  and tangent to  $C_1$ . Let  $C_2$  be the inscribed circle of  $\triangle RST$  and  $C_3$  the inscribed circle of  $\triangle QUV$ . The distance between the centers of  $C_2$  and  $C_3$  can be written as  $\sqrt{10n}$ . What is  $n$ ?

8. A certain function  $f$  has the properties that  $f(3x) = 3f(x)$  for all positive real values of  $x$ , and that  $f(x) = 1 - |x - 2|$  for  $1 \leq x \leq 3$ . Find the smallest  $x$  for which  $f(x) = f(2001)$ .
9. Each unit square of a 3-by-3 unit-square grid is to be colored either blue or red. For each square, either color is equally likely to be used. The probability of obtaining a grid that does not have a 2-by-2 red square is  $m/n$ , where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .
10. How many positive integer multiples of 1001 can be expressed in the form  $10^j - 10^i$ , where  $i$  and  $j$  are integers and  $0 \leq i < j \leq 99$ ?
11. Club Truncator is in a soccer league with six other teams, each of which it plays once. In any of its 6 matches, the probabilities that Club Truncator will win, lose, or tie are each  $1/3$ . The probability that Club Truncator will finish the season with more wins than losses is  $m/n$ , where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .
12. Given a triangle, its *midpoint triangle* is obtained by joining the midpoints of its sides. A sequence of polyhedra  $\mathcal{P}_i$  is defined recursively as follows:  $\mathcal{P}_0$  is a regular tetrahedron whose volume is 1. To obtain  $\mathcal{P}_{i+1}$ , replace the midpoint triangle of every face of  $\mathcal{P}_i$  by an outward-pointing regular tetrahedron that has the midpoint triangle as a face. The volume of  $\mathcal{P}_3$  is  $m/n$ , where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .
13. In quadrilateral  $ABCD$ ,  $\angle BAD \cong \angle ADC$  and  $\angle ABD \cong \angle BCD$ ,  $AB = 8$ ,  $BD = 10$ , and  $BC = 6$ . The length  $CD$  may be written in the form  $m/n$ , where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .
14. There are  $2n$  complex numbers that satisfy both  $z^{28} - z^8 - 1 = 0$  and  $|z| = 1$ . These numbers have the form  $z_m = \cos \theta_m + i \sin \theta_m$ , where  $0 \leq \theta_1 < \theta_2 < \dots < \theta_{2n} < 360$  and angles are measured in degrees. Find the value of  $\theta_2 + \theta_4 + \dots + \theta_{2n}$ .
15. Let  $EFGH$ ,  $EFDC$ , and  $EHBC$  be three adjacent square faces of a cube, for which  $EC = 8$ , and let  $A$  be the eighth vertex of the cube. Let  $I$ ,  $J$ , and  $K$  be points on  $\overline{EF}$ ,  $\overline{EH}$ , and  $\overline{EC}$ , respectively, so that  $EI = EJ = EK = 2$ . A solid  $S$  is obtained by drilling a tunnel through the cube. The sides of the tunnel are planes parallel to  $\overline{AE}$ , and containing the edges  $\overline{IJ}$ ,  $\overline{JK}$ , and  $\overline{KI}$ . The surface area of  $S$ , including the walls of the tunnel, is  $m + n\sqrt{p}$ , where  $m$ ,  $n$ , and  $p$  are positive integers and  $p$  is not divisible by the square of any prime. Find  $m + n + p$ .

Your Exam Manager will have a copy of the 2001 AIME Solution Pamphlet.

WRITE TO US :

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#### 2001 USAMO

THE USA MATHEMATICAL OLYMPIAD (USAMO) is a 6-question, 6-hour, essay-type examination. The USAMO will be held on Tuesday, May 1, 2001. Your teacher has more details on who qualifies for the USAMO in the AMC 10/12 and AIME Teachers' Manuals. The best way to prepare for the USAMO is to study previous years of these exams, the World Olympiad Problems/Solutions and review the contents of the Arbelos. Copies may be ordered as indicated below.

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