AMERICAN MATHEMATICS COMPETITIONS

12th ANNUAL
AMERICAN INVITATIONAL MATHEMATICS EXAMINATION
(AIME)
THURSDAY, MARCH 31, 1994

Sponsored by

Mathematical Association of America
Society of Actuaries Mu Alpha Theta
National Council of Teachers of Mathematics
Casualty Actuarial Society American Statistical Association
American Mathematical Association of Two-Year Colleges
American Mathematical Society

1. DO NOT OPEN THIS BOOKLET UNTIL TOLD TO DO SO BY YOUR PROCTOR.
2. This is a 15-question, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers; i.e., there is neither partial credit nor a penalty for wrong answers.
3. No aids other than scratch paper, graph paper, ruler, compass, and protractor are permitted. In particular, **calculators are not permitted.**
4. Please print the following:

Name: ______________________________
Last   First   Middle initial

Home address: ______________________________
Street Address

____
City   State or Province   Zip or Postcode

____
Home Phone including Area Code   Gender (M or F)   Your age

____
Full Name of School   Grade Level (e.g.,11)

5. Citizenship Status: USA Citizen_____ *Permanent Resident_____ Other_____

   If other, explain: ______________________________

   *Permanent Resident means someone seeking citizenship and currently possessing a U.S.A. Immigration "green card".

7. A combination of the AIME and AHSME scores is used to determine eligibility for participation in the U.S.A. Mathematical Olympiad (USAMO). The USAMO will be given on THURSDAY, April 28, 1994. Please check one box:

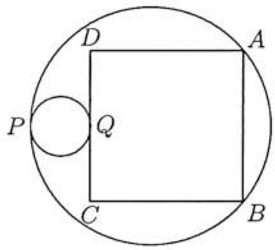
   If I qualify for the USAMO, I agree to take it. YES ☐ NO ☐

   (*Your school must also agree to administer the USAMO before you can take it.*)

8. Record all your answers, and certain other information, on the AIME answer form. Your Examination Manager will instruct you how to complete the form. Only the answer form and this cover will be collected from you.

1. The increasing sequence 3, 15, 24, 48, ... consists of those positive multiples of 3 that are one less than a perfect square. What is the remainder when the 1994^{th} term of the sequence is divided by 1000?

2. A circle with diameter $\overline{PQ}$ of length 10 is internally tangent at P to a circle of radius 20. Square $ABCD$ is constructed with A and B on the larger circle, $\overline{CD}$ tangent at Q to the smaller circle, and the smaller circle outside $ABCD$. The length of $\overline{AB}$ can be written in the form $m+\sqrt{n}$, where m and n are integers. Find $m+n$.



3. The function f has the property that, for each real number x,

$$f(x)+f(x-1)=x^2.$$

If $f(19)=94$, what is the remainder when $f(94)$ is divided by 1000?

4. Find the positive integer n for which

$$\lfloor\log_2 1\rfloor+\lfloor\log_2 2\rfloor+\lfloor\log_2 3\rfloor+\cdots+\lfloor\log_2 n\rfloor=1994.$$

(For real x, $\lfloor x\rfloor$ is the greatest integer $\leq x$.)

5. Given a positive integer n, let $p(n)$ be the product of the non-zero digits of n. (If n has only one digit, then $p(n)$ is equal to that digit.) Let

$$S=p(1)+p(2)+p(3)+\cdots+p(999).$$

What is the largest prime factor of S?

6. The graphs of the equations

$$y=k, \qquad y=\sqrt{3}x+2k, \qquad y=-\sqrt{3}x+2k,$$

are drawn in the coordinate plane for $k=-10,-9,-8,\ldots,9,10$. These 63 lines cut part of the plane into equilateral triangles of side $2/\sqrt{3}$. How many such triangles are formed?

7. For certain ordered pairs (a, b) of real numbers, the system of equations

$$ax + by = 1$$
$$x^2 + y^2 = 50$$

has at least one solution, and each solution is an ordered pair (x, y) of integers. How many such ordered pairs (a, b) are there?

8. The points $(0, 0)$, $(a, 11)$, and $(b, 37)$ are the vertices of an equilateral triangle. Find the value of ab.

9. A solitaire game is played as follows. Six distinct pairs of matched tiles are placed in a bag. The player randomly draws tiles one at a time from the bag and retains them, except that matching tiles are put aside as soon as they appear in the player's hand. The game ends if the player ever holds three tiles, no two of which match; otherwise the drawing continues until the bag is empty. The probability that the bag will be emptied is p/q, where p and q are relatively prime positive integers. Find $p + q$.

10. In triangle ABC, angle C is a right angle and the altitude from C meets $\overline{AB}$ at D. The lengths of the sides of $\triangle ABC$ are integers, $BD = 29^3$, and $\cos B = m/n$, where m and n are relatively prime positive integers. Find $m + n$.

11. Ninety-four bricks, each measuring $4'' \times 10'' \times 19''$, are to be stacked one on top of another to form a tower 94 bricks tall. Each brick can be oriented so it contributes $4''$ or $10''$ or $19''$ to the total height of the tower. How many different tower heights can be achieved using all 94 of the bricks?

12. A fenced, rectangular field measures 24 meters by 52 meters. An agricultural researcher has 1994 meters of fence that can be used for internal fencing to partition the field into congruent, square test plots. The entire field must be partitioned, and the sides of the squares must be parallel to the edges of the field. What is the largest number of square test plots into which the field can be partitioned using all or some of the 1994 meters of fence?

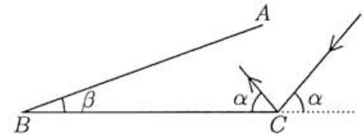
13. The equation

$$x^{10} + (13x - 1)^{10} = 0$$

has 10 complex roots $r_1, \overline{r_1}, r_2, \overline{r_2}, r_3, \overline{r_3}, r_4, \overline{r_4}, r_5, \overline{r_5}$, where the bar denotes complex conjugation. Find the value of

$$\frac{1}{r_1\overline{r_1}} + \frac{1}{r_2\overline{r_2}} + \frac{1}{r_3\overline{r_3}} + \frac{1}{r_4\overline{r_4}} + \frac{1}{r_5\overline{r_5}}.$$

14. A beam of light strikes $\overline{BC}$ at point C with angle of incidence $\alpha = 19.94^\circ$ and reflects with an equal angle of reflection as shown. The light beam continues its path, reflecting off line segments $\overline{AB}$ and $\overline{BC}$ according to the rule: *angle of incidence equals angle of reflection.* Given that $\beta = \alpha/10 = 1.994^\circ$ and $AB = BC$, determine the number of times the light beam will bounce off the two line segments. Include the first reflection at C in your count.



15. Given a point P on a triangular piece of paper ABC, consider the creases that are formed in the paper when A, B, and C are folded onto P. Let us call P a *fold point* of $\triangle ABC$ if these creases, which number three unless P is one of the vertices, do not intersect. Suppose that $AB = 36$, $AC = 72$, and $\angle B = 90^\circ$. Then the area of the set of all fold points of $\triangle ABC$ can be written in the form $q\pi - r\sqrt{s}$, where q, r, and s are positive integers and s is not divisible by the square of any prime. What is $q + r + s$?

SOLUTIONS

A 1994 Solutions Pamphlet will be sent to exam managers within a few weeks.

WRITE TO US!

Correspondence about the problems and solutions for this AIME should be addressed to:

Mr. Richard Parris, AIME Chairman
Phillips Exeter Academy, Exeter, NH 03833 USA

Comments about administrative arrangements and orders for any publications listed below should be addressed to:

Prof. Walter E Mientka, AMC Executive Director
Department of Mathematics and Statistics
University of Nebraska, Lincoln, NE 68588-0658 USA; Phone: 402-472-2257; Fax: 402-472-6087

1994 USAMO

The USA Mathematical Olympiad is a 5-question, $3\frac{1}{2}$ hour, essay-type examination. The USAMO will be held on THURSDAY, April 28, 1994. Your teacher has more details on who qualifies for the USAMO in the AHSME or AIME Teachers' Manuals. The best way to prepare for the USAMO is to study the exams from previous years and to review the contents of the ARBELOS. Copies may be ordered as indicated below.

PUBLICATIONS

MINIMUM ORDER: $5 (before handling fee), US FUNDS ONLY. Canada and US orders must be prepaid. Orders are mailed 4th class, unless you specify 1st class, in which case add $3.00 or 20% of total order, whichever is larger, with a maximum of $15.00. Make checks payable to the American Mathematics Competitions; or give Visa or Mastercard number, expiration date and cardholder's signature.

FOREIGN ORDERS: Do **NOT** prepay; an invoice will be sent.

Examinations: Each price is for **one** copy of an exam **and** its solutions for one year. Specify the years you want and how many copies of each. Prices subject to change after July 1, 1994.

- **AJHSME** (Junior High Exam), 1985-1993, $1 per copy per year.
- **AHSME** 1980-94, $1 per copy per year.
- **AIME** 1983-94, $2 per copy per year.
- USA and International Mathematical **Olympiads** (together), 1976-93, $4 per copy per year.
- National **Summary** of Results and Awards, 1980-93, $ 5 per copy per year.

Books (Exams and solutions):

- Contest Problem Book I, AHSMEs 1950-60, $9.50.
- Contest Problem Book II, AHSMEs 1961-65, $9.50.
- Contest Problem Book III, AHSMEs 1966-72, $11.50.
- Contest Problem Book IV, AHSMEs 1973-82, $11.50.
- USA Mathematical Olympiads, 1972-86, $14.50.
- International Mathematical Olympiads, 1959-77, $11.50.
- International Mathematical Olympiads, 1978-85, $11.50.

Journal

- The ARBELOS (short articles and challenging problems); recommended especially for AIME and USAMO qualifiers. Five volumes plus a Geometry volume, $7.00 each.