AMERICAN MATHEMATICS COMPETITIONS

AIME SOLUTIONS PAMPHLET
FOR STUDENTS AND TEACHERS

12th ANNUAL
AMERICAN INVITATIONAL MATHEMATICS EXAMINATION
(AIME)
THURSDAY, March 31, 1994

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This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs. geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution. *Remember that reproduction of these solutions is prohibited by copyright.*

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1. (Answer: 063)
A positive integer that is one less than a perfect square is of the form

$$n^2 - 1 = (n-1)(n+1),$$

for $n = 2, 3, \ldots$. Such a number is a multiple of 3 if and only if n is not a multiple of 3. Thus the $(2k-1)^{\text{st}}$ and $(2k)^{\text{th}}$ terms of the sequence are $(3k-1)^2 - 1$ and $(3k+1)^2 - 1$ respectively. Therefore, the 1994^{th} term of the sequence is

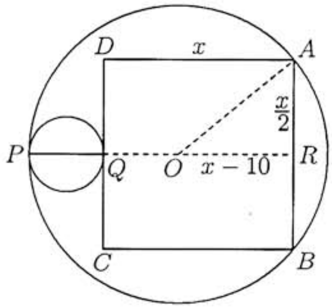
$$(3 \cdot 997 + 1)^2 - 1 = (3000 - 8)^2 - 1 = 3000^2 - 16 \cdot 3000 + 63.$$

When this number is divided by 1000, the remainder is 63.

2. (Answer: 312)
Let O be the center of the large circle. Note that P, Q, and O are collinear since the circles are tangent. Let the line through P, Q, and O intersect $\overline{AB}$ in R and let $x = AB$. Then $RO = RQ - OQ = x - 10$. Because $\overline{CD}$ is tangent at Q to the smaller circle, it follows that $AR = x/2$ and that $\angle ARO$ is a right angle. Hence, by the Pythagorean Theorem,

$$(x-10)^2 + (x/2)^2 = 20^2.$$

Solving for x, we obtain $x = 8 \pm \sqrt{304}$. Since $x > 0$, we have $x = 8 + \sqrt{304}$, and $m + n = 8 + 304 = 312$.



3. (Answer: 561)
Using $f(x) = x^2 - f(x-1)$ repeatedly, we have

$$\begin{aligned} f(94) &= 94^2 - f(93) \\ &= 94^2 - 93^2 + f(92) \\ &= 94^2 - 93^2 + 92^2 - f(91) \\ &\ \ \vdots \\ &= 94^2 - 93^2 + 92^2 - \cdots + 20^2 - f(19) \\ &= (94+93)(94-93) + (92+91)(92-91) + \cdots \\ &\qquad + (22+21)(22-21) + 20^2 - 94 \\ &= (94 + 93 + 92 + \cdots + 21) + 306 \\ &= \frac{94+21}{2} \cdot 74 + 306 \\ &= 4561. \end{aligned}$$

Thus, when $f(94)$ is divided by 1000, the remainder is 561.

4. (Answer: 312)
Let
$$S_n = \lfloor \log_2 1 \rfloor + \lfloor \log_2 2 \rfloor + \lfloor \log_2 3 \rfloor + \cdots + \lfloor \log_2 n \rfloor.$$
Note that, for nonnegative integer k, there are 2^k positive integers x for which $\lfloor \log_2 x \rfloor = k$, namely $x = 2^k, 2^k+1, \ldots, 2^{k+1}-1$. Thus, if r is a positive integer,
$$S_{2^r-1} = 0 + (1+1) + (2+2+2+2) + \cdots + \big(\underbrace{(r-1)+(r-1)+\cdots+(r-1)}_{2^{r-1}\text{ terms}}\big).$$
The right side of this expression has
$$\begin{aligned} 2^r - 2^1 \text{ terms } &\geq 1 \\ 2^r - 2^2 \text{ terms } &\geq 2 \\ 2^r - 2^3 \text{ terms } &\geq 3 \\ &\vdots \\ 2^r - 2^{r-2} \text{ terms } &\geq r-2 \\ 2^r - 2^{r-1} \text{ terms } &= r-1. \end{aligned}$$
It follows that
$$\begin{aligned} S_{2^r-1} &= (2^r - 2^1) + (2^r - 2^2) + (2^r - 2^3) + \cdots + (2^r - 2^{r-1}) \\ &= (r-1)2^r - (2^r - 2) \\ &= (r-2)2^r + 2. \end{aligned}$$
Taking $r = 8$ in this last equation we obtain $S_{255} = 1538 < 1994$. Setting $r = 9$ we find $S_{511} = 3586 > 1994$. Hence, if $S_n = 1994$, then $255 = 2^8 - 1 < n < 2^9 - 1$. Therefore
$$1994 = S_n = S_{255} + (n-255)8 = 8n - 502,$$
and this yields $n = 312$.

5. (Answer: 103)
Consider each positive integer less than 1000 to be a three-digit number by prefixing 0's to numbers with fewer than three digits. The sum of the products of the digits of all such positive numbers is
$$\begin{aligned} &(0\cdot 0\cdot 0 + 0\cdot 0\cdot 1 + 0\cdot 0\cdot 2 + \cdots + 9\cdot 9\cdot 8 + 9\cdot 9\cdot 9) - 0\cdot 0\cdot 0 \\ &\quad = (0+1+2+\cdots+9)^3 - 0. \end{aligned} \tag{$*$}$$
However, $p(n)$ is the product of the non-zero digits of n. The sum of these products can be found by replacing 0 by 1 in the above expression, since ignoring 0's is equivalent

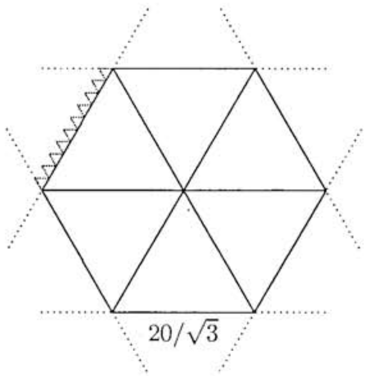
to thinking of them as 1's in the products. (Note that the final 0 in (*) becomes a 1 and compensates for the contribution of 000 after it is changed to 111.) Hence

$$\sum_{n=1}^{999} p(n) = (1+1+2+\cdots+9)^3 - 1 = 46^3 - 1 = (46-1)(46^2+46+1) = 3^3 \cdot 5 \cdot 7 \cdot 103,$$

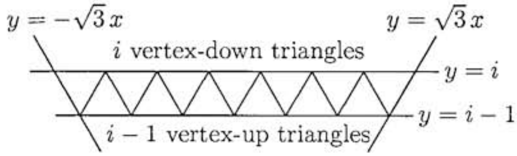
and the largest prime factor is 103.

6. (Answer: 660)

The six outermost lines determine a regular hexagon of side length $20/\sqrt{3}$. The three lines through the origin cut this hexagon into 6 equilateral triangles, each with side length $20/\sqrt{3}$. Since each of these large triangles has side length 10 times the side length of the small triangles, each of the large equilateral triangles is cut into 10^2 small triangles. Hence the hexagon is cut into $6 \cdot 10^2 = 600$ small triangles. In addition there is a row of ten triangles outside, but adjacent to, each side of the hexagon. (Some of these triangles are shown in the figure.) Therefore the total number of small triangles in the configuration is $600 + 6 \cdot 10 = 660$.



Alternate Solution. Let n be a positive integer and consider the figure obtained by drawing the lines for $-n \le k \le n$. By symmetry, the number of equilateral triangles of side $2/\sqrt{3}$ obtained is 6 times the number that lie in the upper half plane in the wedge bounded by the lines $y = \sqrt{3}x$ and $y = -\sqrt{3}x$. Below the line $y = 1$ there is 1 triangle in the wedge. Between $y = 1$ and $y = 2$ there are 3 triangles. For $1 \le i \le n$, there are $2i - 1$ triangles between $y = i - 1$ and $y = i$, and in this row there are $i - 1$ triangles oriented vertex-up and i oriented vertex-down. Above the line $y = n$ there is a row of n vertex-up triangles, but because there is no horizontal line above $y = n$, there are none that are vertex-down. Hence the number of equilateral triangles of side length $2/\sqrt{3}$ is

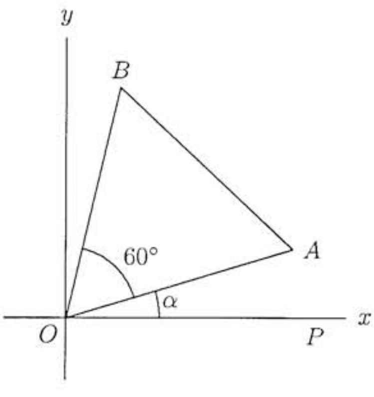


$$6\left((1+3+\cdots+2n-1)+n\right) = 6(n^2+n).$$

When $n = 10$ we have 660 such triangles.

7. (Answer: 072)
The equation $x^2 + y^2 = 50$ is that of the circle with center $(0, 0)$ and radius $5\sqrt{2}$, and $ax + by = 1$ is an equation of a line. The problem statement is equivalent to requiring that the line and the circle intersect and that each intersection be a lattice point. There are 12 lattice points on the circle: $(\pm1, \pm7)$, $(\pm5, \pm5)$, $(\pm7, \pm1)$. Any pair of these points determines a line that intersects the circle in those two points. There are $\binom{12}{2} = 66$ such pairs. Also, at each of the twelve points the tangent line intersects the circle at only that point. Thus, there are 66+12=78 lines that intersect the circle and do so only at lattice points. Any such line can be uniquely written in the form $ax + by = 1$ if and only if the line does not contain the origin. But 6 of the 78 lines *do* contain the origin. These are the lines determined by diametrically opposite points. It follows that there are $78 - 6 = 72$ ordered pairs (a, b) of real numbers for which the given system has at least one solution and has only integer solutions.

8. (Answer: 315)
Let $O = (0, 0)$, $A = (a, 11)$, and $B = (b, 37)$. Note that reflection of the triangle in the y-axis does not change the value of ab. Thus we may assume that the counterclockwise measure of the angle from $\overrightarrow{OA}$ to $\overrightarrow{OB}$ is $60°$.
Let $OB = OA = AB = r$, and let $\angle AOP = \alpha$, where P is a point on the positive x-axis. Then $\angle BOP = \alpha + 60°$. Since



$$\begin{aligned}\sin(\angle BOP) &= \sin(\alpha + 60°)\\ &= \sin\alpha\cos 60° + \cos\alpha\sin 60°,\end{aligned}$$

we have

$$\frac{37}{r} = \frac{11}{r} \cdot \frac{1}{2} + \frac{a}{r} \cdot \frac{\sqrt{3}}{2},$$

from which $a = 21\sqrt{3}$. Similarly

$$\cos(\angle BOP) = \cos(\alpha + 60°) = \cos\alpha\cos 60° - \sin\alpha\sin 60°,$$

which leads to

$$\frac{b}{r} = \frac{a}{r} \cdot \frac{1}{2} - \frac{11}{r} \cdot \frac{\sqrt{3}}{2}.$$

It follows that $b = 5\sqrt{3}$, so $ab = 21\sqrt{3} \cdot 5\sqrt{3} = 315$.

Alternate Solution. Let $O = (0,0)$, $A = (a, 11)$, $B = (b, 37)$, and assume that the counterclockwise measure of the angle from $\overrightarrow{OA}$ to $\overrightarrow{OB}$ is $60°$. Regard A and B as the complex numbers $a + 11i$ and $b + 37i$, respectively. Since a rotation of $60°$ about the origin is equivalent to multiplication by $\cos 60° + i \sin 60°$, we have

$$(a + 11i)(\cos 60° + i \sin 60°) = b + 37i.$$

Separating the real and imaginary parts yields

$$\begin{aligned} a - 11\sqrt{3} &= 2b \\ 11 + a\sqrt{3} &= 74. \end{aligned}$$

From the second equation we obtain $a = 21\sqrt{3}$, and then the first yields $b = 5\sqrt{3}$. Thus $ab = 315$.

9. (Answer: 394)
Let $n \geq 2$ be an integer and let the bag contain n distinct pairs of tiles. The probability that two of the first three tiles selected make a pair is

$$\frac{\text{\# of ways to select three tiles, two of which match}}{\text{\# of ways to select three tiles}} = \frac{n(2n-2)}{\binom{2n}{3}} = \frac{3}{2n-1}.$$

Now let $P(n)$ be the probability of emptying the bag when the bag initially contains n distinct pairs of tiles. Then $P(2) = 1$ and for $n \geq 3$,

$$P(n) = \frac{3}{2n-1} P(n-1).$$

Using this recursion formula repeatedly, we find that

$$P(n) = \frac{3}{2n-1}\frac{3}{2n-3}\cdots\frac{3}{5}P(2).$$

Setting $n = 6$ we have

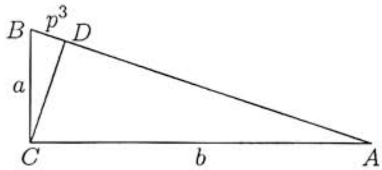
$$P(6) = \frac{3^4}{11 \cdot 9 \cdot 7 \cdot 5} = \frac{9}{385}.$$

The sum of the numerator and denominator is $9 + 385 = 394$.

10. (Answer: 450)

Let a, b, c denote the lengths of $\overline{BC}$, $\overline{AC}$, $\overline{AB}$, respectively. Let $p = 29$, so $BD = p^3$. Since $\triangle BCD \sim \triangle BAC$ we have

$$\frac{p^3}{a} = \frac{a}{c},$$



from which $a^2 = p^3c$. Since p is prime, there is an integer x such that $c = px^2$. It follows that $a = p^2x$. Substituting these expressions for a and c into $b^2 = c^2 - a^2$ we find

$$b^2 = p^2x^4 - p^4x^2 = p^2x^2(x^2 - p^2).$$

Thus, there is a positive integer y with $x^2 - p^2 = y^2$, so $p^2 = x^2 - y^2 = (x-y)(x+y)$. Since p is prime and $x - y < x + y$, we have $x - y = 1$ and $x + y = p^2$, which leads to

$$x = \frac{p^2+1}{2} \quad \text{and} \quad y = \frac{p^2-1}{2}.$$

Since $p = 29$,

$$\cos B = \frac{a}{c} = \frac{p^2x}{px^2} = \frac{p}{(p^2+1)/2} = \frac{29}{(29^2+1)/2}$$

in lowest terms. Hence $m + n = (29^2 + 2 \cdot 29 + 1)/2 = (29+1)^2/2 = 450$.

11. (Answer: 465)

If five bricks in a tower are oriented so that each contributes $10''$ to the height of the tower, then these five bricks contribute $50''$ to the height of the tower. These five bricks can be reoriented so that three contribute $4''$ each and two contribute $19''$ to the height of the tower. Note that with this reorientation, the total height of the tower is unchanged. Thus we may assume that in a tower of bricks there are 0, 1, 2, 3, or 4 bricks oriented so that they each contribute $10''$ to the total height. All other bricks in the tower are oriented to contribute $4''$ or $19''$ to the height.

Suppose that there are y blocks that contribute $10''$ and z blocks that contribute $19''$ to the height of the tower. Then there are $94 - y - z$ bricks that contribute $4''$, and the height of the tower is

$$4(94 - y - z) + 10y + 19z = 376 + 6y + 15z$$

inches. As noted above, we may assume that $0 \le y \le 4$. Because $y + z \le 94$ we must have $0 \le z \le 94 - y$, so there are $95 - y$ possible values for z. In all there are $95+94+93+92+91 = 465$ such pairs (y, z), and hence at most 465 different possible tower heights.

We now show that any two of these ordered pairs give different tower heights. Suppose that ordered pairs (y, z) and (u, v) lead to towers of the same height. Then

$$376 + 6y + 15z = 376 + 6u + 15v.$$

It follows that $6(y-u) = 15(v-z)$, which implies that $y-u$ is divisible by 5. Because $|y-u| \leq 4$ we conclude that $y = u$ and then that $z = v$. Thus the 465 ordered pairs correspond to 465 different tower heights.

Alternate Solution. If we have x $4''$ bricks, y $10''$ bricks and z $19''$ bricks in our tower, then $x + y + z = 94$ and the height of the tower is

$$4x + 10y + 19z = 4(94 - y - z) + 10y + 19z = 376 + 3(2y + 5z).$$

There will be a height corresponding to each possible value of $2y + 5z$ where y and z are non-negative integers with $y + z \leq 94$. With these restrictions, we have

$$2y + 5z \leq 5 \cdot 94 = 470.$$

To count the number of possible values of $2y + 5z$ we apply the following theorem:

> If $\mathrm{GCD}(a, b) = 1$, then every integer greater than or equal to $(a-1)(b-1)$ can be written as $ar + bs$ where r and s are nonnegative integers, and the number of nonnegative integers that cannot be written in this way is $(a-1)(b-1)/2$.

Define an integer n between 0 and 470 to be "good" if n can be written as $2y + 5z$ where $y, z \geq 0$ and $y + z \leq 94$, and "bad" otherwise. By the theorem quoted, there are $(2-1)(5-1)/2 = 2$ bad integers between 0 and 3 (namely 1 and 3). Substituting $z = 94 - x - y$, we see that an integer n between 4 and 470 is good if and only if

$$470 - n = 5x + 3y$$

for some nonnegative integers x and y. Thus by the theorem stated above there are $(3-1)(5-1)/2 = 4$ bad integers between $470 - ((3-1)(5-1) - 1) = 463$ and 470 (namely 463, 466, 468, and 469). Hence the number of good integers and the desired number of tower heights is $471 - 6 = 465$.

Note. The number of heights is equal to the number of terms in the expansion of

$$(x^4 + x^{10} + x^{19})^{94}. \qquad (*)$$

In addition, the coefficient of x^n in the expansion of $(*)$ is the number of different ways of obtaining a tower of height n.

12. (Answer: 702)
Suppose that the field is partitioned into squares of side s. Then there are positive integers m, n with

$$\frac{24}{s} = m \qquad \text{and} \qquad \frac{52}{s} = n.$$

Hence

$$\frac{m}{n} = \frac{6}{13},$$

so there is a positive integer k with $m = 6k$ and $n = 13k$. Note that the total number of test plots is a maximum when k is as large as possible. The total length of fence used in partitioning the field into $s \times s$ squares is

$$(m-1)52 + (n-1)24 = k(6 \cdot 52 + 13 \cdot 24) - (52 + 24) = 624k - 76.$$

Since at most 1994 meters of fence can be used, we have

$$624k - 76 \leq 1994,$$

so that

$$k \leq \frac{1994 + 76}{624} = 3.31\ldots$$

Since k must be an integer, the largest possible value of k is 3. For this value of k we have a total of

$$mn = (6 \cdot 3)(13 \cdot 3) = 702$$

squares, formed by using $624 \cdot 3 - 76 = 1796$ meters of the available 1994 meters of fence.

13. (Answer: 850)
Let $p(x) = x^{10} + (13x - 1)^{10}$. If r is a zero of $p(x)$, then

$$-1 = \left(\frac{13r - 1}{r}\right)^{10} = \left(\frac{1}{r} - 13\right)^{10}.$$

Thus

$$\left(\frac{1}{r} - 13\right)\left(\frac{1}{\overline{r}} - 13\right) = 1,$$

so that

$$\left(\frac{1}{r_1} - 13\right)\left(\frac{1}{\overline{r_1}} - 13\right) + \cdots + \left(\frac{1}{r_5} - 13\right)\left(\frac{1}{\overline{r_5}} - 13\right) = 5.$$

Expanding and rearranging, we find

$$\left(\frac{1}{r_1\overline{r_1}} + \cdots + \frac{1}{r_5\overline{r_5}}\right) - 13\left(\frac{1}{r_1} + \frac{1}{\overline{r_1}} + \cdots + \frac{1}{r_5} + \frac{1}{\overline{r_5}}\right) + 5 \cdot 169 = 5.$$

Note that $1/r_1$, $1/\overline{r_1}, \ldots, 1/r_5$, $1/\overline{r_5}$ are the zeros of

$$x^{10}p\left(\frac{1}{x}\right) = x^{10} - 130x^9 + \cdots,$$

so

$$\frac{1}{r_1} + \frac{1}{\overline{r_1}} + \cdots + \frac{1}{r_5} + \frac{1}{\overline{r_5}} = 130.$$

Therefore,

$$\frac{1}{r_1\overline{r_1}} + \cdots + \frac{1}{r_5\overline{r_5}} = 13 \cdot 130 - 5 \cdot 169 + 5 = 850.$$

Alternate Solution. Let $p(x) = x^{10} + (13x-1)^{10}$. If $p(r) = 0$, then

$$\left(13 - \frac{1}{r}\right)^{10} = -1 = \cos 180^\circ + i \sin 180^\circ.$$

It follows that

$$\frac{1}{r} = 13 - (\cos\theta + i\sin\theta)$$

where θ is an odd multiple of 18°. Hence

$$\frac{1}{r\overline{r}} = (13 - (\cos\theta + i\sin\theta))\,(13 - (\cos\theta - i\sin\theta)) = 170 - 26\cos\theta.$$

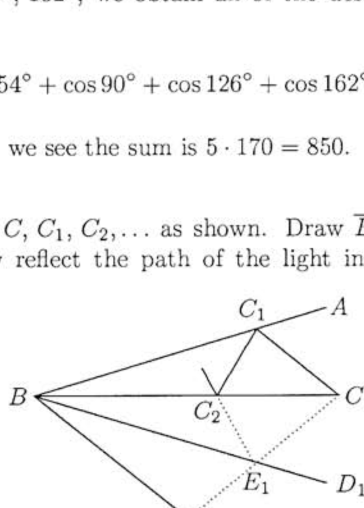
Letting θ take on the values 18°, 54°, 90°, 126°, 162°, we obtain all of the desired products. Thus

$$\frac{1}{r_1\overline{r_1}} + \cdots + \frac{1}{r_5\overline{r_5}} = 5 \cdot 170 - 26(\cos 18^\circ + \cos 54^\circ + \cos 90^\circ + \cos 126^\circ + \cos 162^\circ).$$

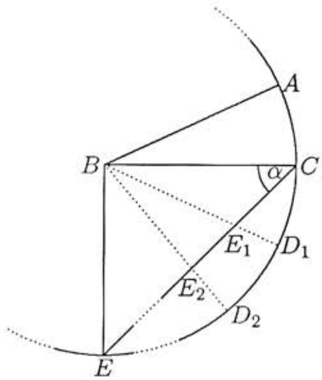
Applying the identity $\cos\theta + \cos(180^\circ - \theta) = 0$, we see the sum is $5 \cdot 170 = 850$.

14. (Answer: 071)

Label the points at which the light reflects as C, C_1, $C_2, \ldots$ as shown. Draw $\overline{BD_1}$ so that $BD_1 = BC$ and $\angle CBD_1 = \beta$. Now reflect the path of the light inside $\angle ABC$ across $\overline{BC}$, and let E_1 be the reflection of C_1. For counting purposes, it doesn't matter whether we look at the real path $CC_1C_2\ldots$ or the reflected path beginning with $\overline{CE_1}$, so assume that the light beam actually begins its path by travelling from C to E_1. Now draw $\overline{BD_2}$ with $\angle D_1BD_2 = \beta$ and $BD_2 = BD_1$. Reflect the new path of the light beam across $\overline{BD_1}$ and let the reflection of C_2 be E_2. Since $\angle CE_1D_1 = \angle C_2E_1B = \angle BE_1E_2$, the path CE_1E_2 must be a straight line.



Repeat the above process by constructing $\overline{BD_3}$, $\overline{BD_4}\ldots$ with $\angle D_iBD_{i+1} = \beta$ and $BD_i = BC$. The result is a new path which follows the ray $\overrightarrow{CE_1}$. Each point where the light beam reflects off $\overline{BA}$ or $\overline{BC}$ will correspond to an intersection point E_i of this ray with some $\overline{BD_i}$. We need to count these intersections. Draw the circle with center B and radius AB as shown. The path that concerns us is the line segment $\overline{CE}$ where E is on the circle and $\angle BCE = \alpha$. (Once the ray leaves the circle, there will be no more reflections in the original path since the light beam will be outside $\triangle ABC$.) To count the number of intersections E_i first find $\angle CBE$. Since $BE = BC$, we have $\angle BEC = \angle BCE$, so $\angle CBE = 180 - 2\alpha$. Hence the number of intersections equals

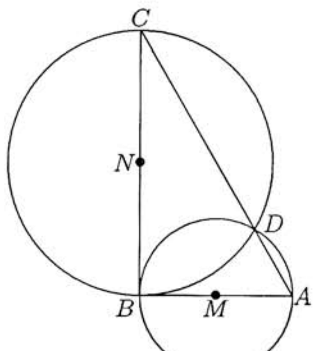


$$\left\lfloor \frac{180 - 2\alpha}{\beta} \right\rfloor + 1,$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to x, and we add 1 to count the first reflection at C. With $\alpha = 19.94$ and $\beta = 1.994$, we obtain 71 reflections.

15. (Answer: 597)

We claim that the set of all fold points of $\triangle ABC$ is the region common to the interiors of the circles that have $\overline{AB}$ and $\overline{BC}$ as their diameters. To establish this, first show that the creases formed by folding vertices A and B onto P are disjoint if and only if P lies inside the circle that has $\overline{AB}$ as diameter. To see this, note that the crease formed by folding any point Q onto P is part of the perpendicular bisector of $\overline{QP}$. If P is outside the circle with diameter $\overline{AB}$, then $\triangle PAB$ is not obtuse, for $\angle P$ is acute, and angles PAB and PBA are at most 60° and 90° respectively. Therefore the circumcenter of $\triangle PAB$ is inside the triangle, requiring that the creases intersect. On the other hand, if P is inside the circle that has diameter $\overline{AB}$, then the perpendicular bisectors of $\overline{PA}$ and $\overline{PB}$ meet at a point that is separated from P by $\overline{AB}$, so the creases do not meet inside $\triangle ABC$. If P is on the circle, then the creases meet on $\overline{AB}$. A similar discussion applies to the circle that has $\overline{BC}$ as diameter and to the circle that has $\overline{AC}$ as diameter. Note, however, that all interior points of $\triangle ABC$ are inside the latter circle. Thus the set of fold points of the triangle is the region common to the interior of the triangle and the interiors of the two circles with diameters $\overline{AB}$ and $\overline{BC}$.



These two circles both intersect $\overline{AC}$ at D, the foot of the perpendicular from B to $\overline{AC}$. The region in question is therefore bounded by two circular arcs. One is a 120° arc of radius 18, centered at M, the midpoint of $\overline{AB}$; the other is a 60° arc of radius $18\sqrt{3}$, centered at N, the midpoint of $\overline{BC}$. The areas of triangles DMB and DNB are $\frac{1}{2}\cdot 18\cdot 18\cdot \frac{\sqrt{3}}{2} = 81\sqrt{3}$ and $\frac{1}{2}\cdot 18\sqrt{3}\cdot 18\sqrt{3}\cdot \frac{\sqrt{3}}{2} = 243\sqrt{3}$, respectively. The areas of sectors DMB and DNB are $\frac{1}{3}\pi\cdot 18^2 = 108\pi$ and $\frac{1}{6}\pi(18\sqrt{3})^2 = 162\pi$, respectively. The area that we seek is the sum of the sectors' areas minus the sum of the triangles' areas, which simplifies to $270\pi - 324\sqrt{3}$. The desired sum is $q+r+s = 270+324+3 = 597$.