1. (Answer: 007)

Use the identity $\log_b a = \dfrac{1}{\log_a b}$ for a and b greater than 0 to obtain

$$\begin{aligned}\frac{2}{\log_4 2000^6} + \frac{3}{\log_5 2000^6} &= 2\log_{2000^6} 4 + 3\log_{2000^6} 5\\ &= \log_{2000^6}(4^2 \cdot 5^3) = \log_{2000^6} 2000\\ &= \frac{1}{\log_{2000} 2000^6} = \frac{1}{6}\,.\end{aligned}$$

2. (Answer: 098)

In addition to the intercepts (2000,0) and $(-2000, 0)$, there are an equal number of lattice points in each quadrant on the hyperbola. Assume therefore that x and y are positive integers. Factor the difference of squares to obtain $(x-y)(x+y) = 2000^2$, which shows that $x-y$ and $x+y$ must be divisors of 2000^2. Notice that $x-y$ and $x+y$ must have the same parity because their sum is even. Because their product is also even, each of $x-y$ and $x+y$ must be even. Thus there are positive integers m and n such that $x-y = 2m$ and $x+y = 2n$, where $mn = 1000^2$ and $m < n$. Because $1000^2 = 2^6 5^6$ has $7 \cdot 7 = 49$ divisors, it follows that there are 24 such pairs (m, n), hence 24 first-quadrant lattice points on the hyperbola. In all, there are $2 + 4 \cdot 24 = 98$ lattice points on the hyperbola.

3. (Answer: 758)

The 38 cards remaining after the first pair is removed consist of nine sets of four cards with equal numbers, plus one pair. The total number of possible two-card sets for the second drawing is $\binom{38}{2}$. The total number of possible *pairs* for the second drawing is $9\binom{4}{2} + 1$, because there are nine ways to select a four-card set and $\binom{4}{2}$ ways to select two cards from that set, plus one way to select the remaining pair. Thus the probability that the second drawing produces a pair is

$$\frac{9\binom{4}{2}+1}{\binom{38}{2}} = \frac{9 \cdot 6 + 1}{703} = \frac{55}{703},$$

and $m + n = 758$.

4. (Answer: 180)
For any positive integer x, let

$$x = 2^{a_1} p_2^{a_2} p_3^{a_3} \cdots p_n^{a_n}$$

be the prime decomposition of x, where $0 \le a_1$ and $1 \le a_i$ for $1 < i$, and where the p_i are primes, with $2 < p_2 < p_3 < \ldots < p_n$. If x has the required number of divisors, then

$$6 = \prod_{i=2}^{n} (a_i + 1) \quad \text{and} \quad 12 = a_1 \prod_{i=2}^{n} (a_i + 1),$$

which implies that $a_1 = 2$. It also follows that either $n = 2$ and $a_2 = 5$, or else $n = 3$ and $\{a_2, a_3\} = \{1, 2\}$. Thus $x = 2^2 p_2^5$, or $x = 2^2 p_2^2 p_3$, or $x = 2^2 p_2 p_3^2$. To find the smallest possible x, test the smallest possible values for p_2 and p_3. This gives the three candidates $2^2 \cdot 3^5 = 972$, $2^2 \cdot 3^2 \cdot 5 = 180$, and $2^2 \cdot 3 \cdot 5^2 = 300$, among which $x = 180$ is minimal.

5. (Answer: 376)
There are $\binom{8}{5}$ ways to select the rings to be worn. There are $\binom{8}{5}$ ways to assign 5 indistinguishable rings to 4 fingers, which can be seen by inserting 3 separators into a row of five rings, to designate the spaces between the four fingers. For each assignment of 5 indistinguishable rings, there are 5! assignments of distinguishable rings. Thus

$$N = \binom{8}{5} \cdot \binom{8}{5} \cdot 5! = 376320.$$

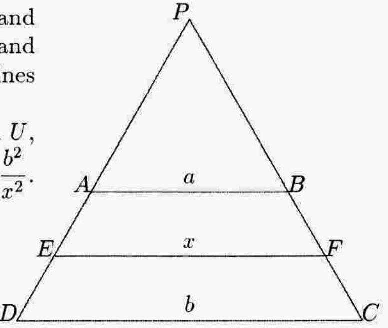
6. (Answer: 181)
Let the bases of the trapezoid be a and $a+100$, and let the height be h. Then the *median* — the segment that joins the midpoints of the other two sides — has length $a+50$, and the area of the trapezoid is $h(a+50)$. The median divides the trapezoid into two trapezoids, the smaller of which has area

$$\frac{2}{5}(a+50)h = (a+25)\frac{h}{2}.$$

Solve this equation to find that $a = 75$.

The requested length is given by the formula $x^2 = \frac{a^2+b^2}{2}$ (where a and b are the bases of the trapezoid) which is proved below. Substitute $a = 75$ and $b = 175$ to find that $x^2 = 18125$, hence that $x^2/100 = 181.25$.

It remains to prove the formula. Assume $a < b$, and let x be the length of a segment that joins the legs and that is parallel to the bases of the trapezoid. The lines containing the legs intersect at point P. Denote the areas of PAB, $ABFE$, and $EFCD$ as S, T and U, respectively. Then $\frac{S}{S+T} = \frac{a^2}{x^2}$ and $\frac{S+T+U}{S+T} = \frac{b^2}{x^2}$. Thus, $\frac{a^2}{x^2} + \frac{b^2}{x^2} = \frac{2S+T+U}{S+T}$. When $T = U$, $\frac{a^2}{x^2} + \frac{b^2}{x^2} = 2$, from which we obtain the formula.



7. (Answer: 137)
Multiply both sides of the given equation by 19! to obtain

$$\binom{19}{2}+\binom{19}{3}+\binom{19}{4}+\binom{19}{5}+\binom{19}{6}+\binom{19}{7}+\binom{19}{8}+\binom{19}{9}=19N$$

The identity

$$\sum_{k=0}^{19}\binom{19}{k}=2^{19}$$

yields

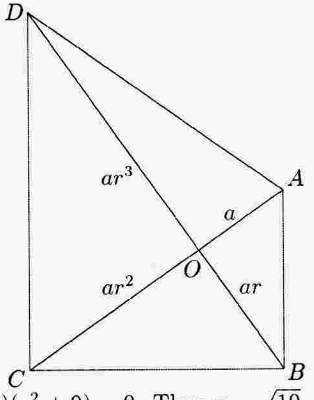
$$\sum_{k=0}^{9}\binom{19}{k}=2^{18}.$$

Thus, $19N=2^{18}-19-1=262124$. Dividing 2621 by 19, we obtain an integer part of 137.

8. (Answer: 110)
Let O be the intersection of the diagonals. By considering the altitudes drawn to the hypotenuses of right triangles ABC and BCD, we obtain

$$\frac{OA}{OB}=\frac{OB}{OC} \quad \text{and} \quad \frac{OB}{OC}=\frac{OC}{OD},$$

so that OA, OB, OC, and OD form a geometric progression. Let us represent them by a, ar, ar^2, and ar^3, respectively. Apply the Pythagorean Theorem to triangles AOB and AOD to obtain $AB=a\sqrt{1+r^2}$ and $AD=a\sqrt{1+r^6}$. It follows that



$$91=\frac{1001}{11}=\frac{AD^2}{AB^2}=\frac{1+r^6}{1+r^2}=r^4-r^2+1,$$

or $r^4-r^2-90=0$, which can be factored as $(r^2-10)(r^2+9)=0$. Thus $r=\sqrt{10}$. From

$$AB=\sqrt{11}=a\sqrt{1+r^2},$$

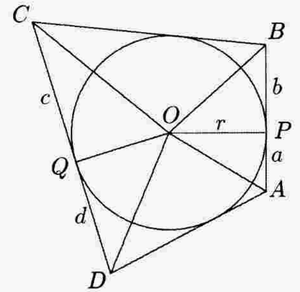
we obtain $a=1$. Thus $BC^2=OB^2+OC^2=10+100=110$.

9. (Answer: 000)

Let $z = r(\cos\theta + i\sin\theta)$. Then $\frac{1}{z} = r(\cos\theta - i\sin\theta)$, and $z + \frac{1}{z} = 2r\cos\theta$. Thus, $z = \cos 3^\circ \pm i\sin 3^\circ$ and $\frac{1}{z} = \cos 3^\circ \mp i\sin 3^\circ$. Now $z^{2000} + \frac{1}{z^{2000}} = 2\cos 6000^\circ = 2\cos 240^\circ = -1$. The least integer greater than -1 is 0.

10. (Answer: 647)

Let O be the center of the circle. As shown in the figure below, let $AP = a$, $PB = b$, $CQ = c$, $QD = d$, and $OQ = r = OP$. Because the circle is inscribed in $ABCD$, the radial segments $\overline{OA}$, $\overline{OB}$, $\overline{OC}$, and $\overline{OD}$ bisect the angles of the quadrilateral. Thus let 2α, 2β, 2γ, and 2δ be the measures of angles DAB, ABC, BCD, and CDA, respectively, and note that $\tan\alpha = r/a$, $\tan\beta = r/b$, $\tan\gamma = r/c$, and $\tan\delta = r/d$. The relation $2\alpha + 2\beta + 2\gamma + 2\delta = 360$ implies that $\tan(\alpha+\beta) + \tan(\gamma+\delta) = 0$. Apply the formula for the tangent of a sum twice to obtain



$$\frac{\frac{r}{a} + \frac{r}{b}}{1 - \frac{r^2}{ab}} + \frac{\frac{r}{c} + \frac{r}{d}}{1 - \frac{r^2}{cd}} = 0.$$

This equation can be rewritten in the form

$$r^2 = \frac{ab(c+d) + cd(a+b)}{a+b+c+d}.$$

Substitute the given values of a, b, c, and d into this equation to find that $r^2 = 647$.

11. (Answer: 131)

A point whose coordinates are integers is called a *lattice point*. The translation that maps D to (0,0) produces a trapezoid $A'B'C'D'$ whose vertices are also lattice points, and whose sides have the same slopes as those of $ABCD$. The translation that maps C' to D' maps B' to a lattice point B'' that makes $A'D' = D'B''$. Because this process is reversible, we need only find the lattice points P for which $D'P = D'A'$. These points are found on the circle whose equation is $x^2 + y^2 = 50$, and there are twelve of them: $P_1 = (-1,-7)$, $P_2 = (1,-7)$, $P_3 = (5,-5)$, $P_4 = (7,-1)$, $P_5 = (7,1)$, $P_6 = (5,5)$, $P_7 = (1,7)$, $P_8 = (-1,7)$, $P_9 = (-5,5)$, $P_{10} = (-7,1)$, $P_{11} = (-7,-1)$, and $P_{12} = (-5,-5)$. Choosing $B'' = P_1$ makes $A'B'C'D'$ a parallelogram, choosing $B'' = P_2$ or $B'' = P_8$ makes $\overline{A'B'}$ horizontal or vertical, and choosing $B'' = P_7$ makes $A'B'C'D'$ into a segment. In the other eight cases, the slope of $\overline{P_1P_k}$ is the same as the slope of $\overline{AB}$. These slopes are $1/3$, $3/4$, 1, 2, -3, $-4/3$, -1, and $-1/2$. The sum of their absolute values is $119/12$, so $m + n = 131$.

12. (Answer: 118)

Let P be the point of intersection of the line through O that is perpendicular to the plane of A, B and C. Then triangles OPA, OPB and OPC are congruent, and P is equidistant from A, B and C. Therefore, P is the circumcemter of triangle ABC. Thus, $OP^2 = AO^2 - AP^2 = 20^2 - R^2$, where R is the circumradius of triangle ABC. From the Extended Law of Sines, we have $2R = \dfrac{a}{\sin A} = \dfrac{abc}{bc\sin A} = \dfrac{abc}{2K}$, where K is the area of triangle ABC. Using Heron's Formula, we find the area of triangle ABC to be $\sqrt{21 \cdot 8 \cdot 7 \cdot 6} = 84$. Apply this result to obtain $R = \dfrac{abc}{4K} = \dfrac{13 \cdot 14 \cdot 15}{4 \cdot 84} = \dfrac{65}{8}$. Then $OP^2 = 20^2 - \left(\dfrac{65}{8}\right)^2 = \dfrac{160^2 - 65^2}{8^2} = \dfrac{(160+65)(160-65)}{8^2}$, and $OP = \dfrac{15\sqrt{95}}{8}$.

13. (Answer: 200)

Divide both sides of the equation by x^3 and regroup to obtain

$$2\left(1000x^3 - \frac{1}{x^3}\right) + \left(100x^2 + \frac{1}{x^2}\right) + 10 = 0. \qquad (*)$$

Now let $u = 10x - \frac{1}{x}$. Then

$$\begin{aligned} u^2 &= 100x^2 + \frac{1}{x^2} - 20, \text{ and} \\ u^3 &= 1000x^3 - \frac{1}{x^3} - 30\left(10x - \frac{1}{x}\right) \\ &= 1000x^3 - \frac{1}{x^3} - 30u. \end{aligned}$$

The equation (*) can be written as $2(u^3 + 30u) + u^2 + 20 + 10 = 0$. This simplifies to $2u^3 + u^2 + 60u + 30 = 0$ which can be expressed as $u^2(2u+1) + 30(2u+1) = 0$ or $(u^2+30)(2u+1) = 0$. When x is real, u must be too, so $10x - (1/x) = -1/2$. Solving, we obtain $x = \dfrac{-1 \pm \sqrt{161}}{40}$, so $m + n + r = 200$.

14. (Answer: 495)
Because $(n+1)! - n! = n!(n+1) - n! = n!n$, it follows that

$$\begin{aligned}(n+16)! - n! &= (n+16)! - (n+15)! + (n+15)! - (n+14)! + \cdots + (n+1)! - n! \\ &= (n+15)!(n+15) + (n+14)!(n+14) + \cdots + (n+1)!(n+1) + n!n.\end{aligned}$$

This shows that the factorial base expansion of $(n+16)! - n!$ is

$$(0, 0, \ldots, 0, n, n+1, \ldots, n+14, n+15),$$

which begins with a block of $n-1$ zeros. The factorial base expansion of 16! is $(0, 0, \ldots, 0, 1)$, so the requested expansion is

$$(0, 0, \ldots, 0, 1, 0, \ldots, 0, 32, 33, \ldots, 47, 0, \ldots, 0, 64, \ldots, 79, 0, \ldots, 0, \ldots, 1984, \ldots, 1999).$$

Notice that, starting in position thirty-two, the expansion contains groups of sixteen nonzeros alternating with groups of sixteeen zeros. With the exception of $f_{16} = 1$, each nonzero f_i is i. Each of the 62 groups of sixteen nonzeros contributes 8 to the alternating sum, and f_{16} contributes -1, so the requested value is $8 \cdot 62 - 1 = 495$.

15. (Answer: 001)
Multiply both sides of the given equation by $\sin 1°$, and note that $\sin 1° = \sin[(x+1)-x]° = \sin(x+1)° \cos x° - \cos(x+1)° \sin x°$. Thus

$$\frac{\sin 1°}{\sin x° \sin(x+1)°} = \frac{\cos x° \sin(x+1)° - \sin x° \cos(x+1)°}{\sin x° \sin(x+1)°} = \cot x° - \cot(x+1)°$$

Now we have

$$\begin{aligned}\frac{\sin 1°}{\sin n°} &= (\cot 45° - \cot 46°) + (\cot 47° - \cot 48°) + \ldots + (\cot 133° - \cot 134°) \\ &= \cot 45° - (\cot 46° + \cot 134°) + (\cot 47° + \cot 133°) - \ldots + (\cot 89° + \cot 91°) - \cot 90° \\ &= \cot 45° = 1.\end{aligned}$$

Therefore $\sin n° = \sin 1°$, and the least positive integer value for n is 1.