AMERICAN MATHEMATICS COMPETITIONS

AIME SOLUTIONS PAMPHLET
FOR STUDENTS AND TEACHERS

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(AIME)

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This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs. geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution. *Remember that reproduction of these solutions is prohibited by copyright.*

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1. (Answer: 200)
The sum of the entries in the first row, which is $x+115$, equals the sum of the entries in the first column, hence the lower-left entry is 114. Because the sum of the entries in the diagonal that includes the upper-right corner is $x+115$, the central entry must be $x-95$. Because the sum of the entries in the second row is $x+115$, the last entry in that row must be 209. Because the sum of the entries in the third column is $x+115$, the last entry in that column must be $x-190$. This puts x, $x-95$, and $x-190$ on a diagonal. It follows that

$$x+(x-95)+(x-190)=x+115,$$

hence $x=200$.

2. (Answer: 340)
Because

$$\lfloor \log_2 n \rfloor = k \Longleftrightarrow 2^k \le n < 2^{k+1},$$

in order for the integer k to be positive and even,

$$n \in \{\underbrace{4,5,6,7}_{4},\underbrace{16,17,\ldots,31}_{16},\underbrace{64,65,\ldots,127}_{64},\underbrace{256,257,\ldots,511}_{256}\},$$

so there are $4+16+64+256=340$ possible choices for n.

3. (Answer: 044)
Notice that

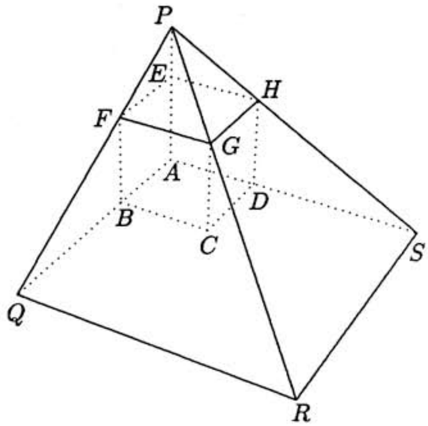
$$(xy-3x+7y-21)^n=(x+7)^n(y-3)^n.$$

The simplified expansions of $(x+7)^n$ and $(y-3)^n$ have $n+1$ terms each. When these two expansions are multiplied together, $(n+1)^2$ terms of the form cx^jy^k are produced. No two of these terms are like terms because they differ in at least one exponent. Hence the expansion of $(xy-3y+7y-21)^n$ has $(n+1)^2$ terms. To make $(n+1)^2 \ge 1996$, we need $n \ge \sqrt{1996}-1$. The smallest such integer n is 44.

4. (Answer: 166)

Let $ABCDEFGH$ be the cube, P be the point source of light, $PE = x$, and $EA = 1$. In the diagram at right, P, E, and A are collinear, and Q, R, and S are the intersections of the extensions of $\overline{PF}$, $\overline{PG}$, and $\overline{PH}$, respectively, with the plane of $ABCD$. Because the squares $EFGH$ and $ABCD$ are in parallel planes, it follows that pyramids $PEFGH$ and $PAQRS$ are similar. Therefore $AQRS$ is a square, and

$$\frac{AQ}{EF} = \frac{AP}{EP}.$$



Solve this equation to obtain $AQ = \dfrac{x+1}{x}$. The area of the shadow is 48, hence

$$\left(\frac{x+1}{x}\right)^2 - 1 = 48.$$

Thus $x = 1/6$ and $1000\,x = 166\frac{2}{3}$.

5. (Answer: 023)

The first equation implies that $a+b+c=-3$. The second equation implies that $t=-(a+b)(b+c)(c+a)$. It follows that $t=-(-3-c)(-3-a)(-3-b)$, which expands to $t=27+9(a+b+c)+3(ab+bc+ca)+abc$. The first equation implies that $ab+bc+ca=4$ and $abc=11$, hence that $t=27-27+12+11=23$.

OR

The first equation implies that $a+b+c=-3$. It follows that the roots of the second equation are $-3-c$, $-3-a$, and $-3-b$. These are also the roots of the equation $(-x-3)^3+3(-x-3)^2+4(-x-3)-11=0$, obtained by replacing x by $-x-3$ in the first equation. The leading coefficient of this equation is -1 and the constant term is $(-3)^3+3(-3)^2+4(-3)-11=-23$; thus $t=23$.

6. (Answer: 049)
The five teams must play a total of $5 \cdot 4/2 = 10$ games, so there are $2^{10} = 1024$ possible outcomes for the tournament. Team A wins all four of its games in $2^{10-4} = 64$ of these outcomes. Because at most one team can be undefeated, there are $5 \cdot 64 = 320$ tournaments that produce an undefeated team. A similar argument shows that 320 of the 1024 possible tournaments produce a winless team. These possibilities are not mutually exclusive, however. In $2^{10-7} = 8$ of the tournaments, team A is undefeated and team B is winless, and there are $5 \cdot 4 = 20$ such two-team permutations. In other words, $8 \cdot 20 = 160$ of the 1024 tournaments have both an undefeated team and a winless team. Thus, according to the inclusion-exclusion principle, there are $1024 - 320 - 320 + 160 = 544$ tournament outcomes in which there is neither an undefeated nor a winless team. All outcomes are equally likely, hence the required probability is $544/1024 = 17/32$, and $17 + 32 = 49$.

7. (Answer: 300)
There are $\binom{49}{2} = 1176$ ways to select the positions of the yellow squares. Because quarter-turns can be applied to the board, however, there are fewer than 1176 inequivalent color schemes. Color schemes in which the two yellow squares are *not* diametrically opposed appear in four equivalent forms. Color schemes in which the two yellow squares *are* diametrically opposed appear in two equivalent forms, and there are $(49 - 1)/2 = 24$ such pairs of yellow squares. Thus the number of inequivalent color schemes is

$$\frac{1176 - 24}{4} + \frac{24}{2} = 300.$$

8. (Answer: 799)
Let $n = 6^{20}$. Suppose that x and y are positive integers for which

$$n = \frac{1}{\frac{1}{2}\left(\frac{1}{x} + \frac{1}{y}\right)} = \frac{2xy}{x + y}.$$

It follows that $xy - \frac{n}{2}x - \frac{n}{2}y = 0$, hence that

$$\left(x - \frac{n}{2}\right)\left(y - \frac{n}{2}\right) = \frac{n^2}{4} = \frac{6^{40}}{4} = 2^{38}3^{40}.$$

Because $2^{38}3^{40}$ has $39 \cdot 41 = 1599$ positive divisors, there are $\frac{1598}{2} = 799$ pairs of unequal positive integers whose product is $2^{38}3^{40}$, and therefore 799 ordered pairs (x, y) of the required type.

9. (Answer: 342)
Suppose that there are 2^k lockers in the row, and let L_k be the number of the last locker opened, once all the lockers are open. After the student makes his first pass along the row, there are 2^{k-1} closed lockers left. These closed lockers all have even numbers and are in descending order from where the student is standing. Now renumber the closed lockers from 1 to 2^{k-1}, starting from the end where the student is standing. Notice that the locker originally numbered n (where n is even) is now numbered $2^{k-1}+1-n/2$. Thus, because L_{k-1} is the number of the last locker opened with this new numbering, we have

$$L_{k-1} = 2^{k-1} + 1 - \frac{L_k}{2}.$$

Solving for L_k we find

$$L_k = 2^k + 2 - 2L_{k-1}.$$

Iterate this recursion once to obtain

$$L_k = 2^k + 2 - 2(2^{k-1} + 2 - 2L_{k-2}) = 4L_{k-2} - 2. \qquad (1)$$

When there are $1024 = 2^{10}$ lockers to start with, the last locker to be opened is numbered L_{10}. Apply (1) repeatedly to $L_0 = 1$ to find that $L_2 = 4L_0 - 2 = 2$, $L_4 = 6$, $L_6 = 22$, $L_8 = 86$, and $L_{10} = 342$.

OR

Follow the given solution to the recursion (1), which can be written in the form

$$L_k - \frac{2}{3} = 4\left(L_{k-2} - \frac{2}{3}\right).$$

Because $L_0 = 1$ and $L_1 = 2$, it follows that

$$L_k - \frac{2}{3} = \begin{cases} \left(1 - \frac{2}{3}\right) 4^{k/2}, & \text{if } k \text{ is even,} \\ \left(2 - \frac{2}{3}\right) 4^{(k-1)/2}, & \text{if } k \text{ is odd.} \end{cases}$$

These formulas may be combined to yield

$$L_k = \frac{1}{3}\left(4^{\lfloor (k+1)/2 \rfloor} + 2\right)$$

for all nonnegative k. In particular, $L_{10} = 342$.

Query: How would the solution change if there were 1000 lockers in the hall?

10. (Answer: 159)
The identity

$$\frac{\cos A + \sin A}{\cos A - \sin A} = \frac{1 + \tan A}{1 - \tan A} = \frac{\tan 45^\circ + \tan A}{1 - \tan 45^\circ \tan A} = \tan(45^\circ + A)$$

implies that the given equation is equivalent to $\tan 19x^\circ = \tan(45^\circ + 96^\circ) = \tan 141^\circ$. It follows that $19x$ must differ from 141 by a multiple of 180; that is,

$$19x = 141 + 180y = 19(7 + 9y) + (8 + 9y),$$

for some integer y. The smallest positive x corresponds to the smallest nonnegative y for which $8 + 9y = 19z$ for some positive integer z. Solve for y to obtain $y = 2z + \frac{z-8}{9}$, from which it follows that the minimum value for z is 8. Hence $y = 16$ and $x = 159$.

OR

Because $\sin 96^\circ = \cos 6^\circ$, the given equation is equivalent to

$$\tan 19x^\circ = \frac{\cos 96^\circ + \cos 6^\circ}{\cos 96^\circ - \cos 6^\circ}.$$

The identities

$$\cos(A + B) + \cos(A - B) = 2 \cos A \cos B$$

and

$$\cos(A + B) - \cos(A - B) = -2 \sin A \sin B$$

imply that

$$\frac{\cos 96^\circ + \cos 6^\circ}{\cos 96^\circ - \cos 6^\circ} = \frac{2 \cos 51^\circ \cos 45^\circ}{-2 \sin 51^\circ \sin 45^\circ} = -\frac{\sin 39^\circ}{\cos 39^\circ}.$$

It follows that the given equation is equivalent to

$$\tan 19x^\circ = -\tan 39^\circ = \tan 141^\circ.$$

The solution continues as above.

OR

Notice that $A\cos x + B\sin x$ is equivalent to $C\cos(x-\phi)$, where $C^2 = A^2 + B^2$, $A = C\cos\phi$, and $B = C\sin\phi$. Hence the given equation is equivalent to

$$\tan 19x^\circ = \frac{\sqrt{2}\cos(96^\circ - 45^\circ)}{\sqrt{2}\cos(96^\circ + 45^\circ)} = \frac{\cos 51^\circ}{\cos 141^\circ} = \frac{\sin 141^\circ}{\cos 141^\circ} = \tan 141^\circ.$$

The solution continues as before.

11. (Answer: 276)
Divide both sides of the given equation by z^3, which gives $z^3 + z + 1 + z^{-1} + z^{-3} = 0$. This takes the form $w^3 - 2w + 1 = 0$, where $w = z + z^{-1}$. Factor the cubic polynomial to obtain $(w-1)(w^2+w-1) = 0$. Now replace w by $z + z^{-1}$ and multiply both sides of the equation by z^3. This yields

$$0 = (z^2 - z + 1)(z^4 + z^3 + z^2 + z + 1) = \frac{z^3+1}{z+1}\cdot\frac{z^5-1}{z-1}.$$

It follows that the six values for z are the fifth roots of 1 and the cube roots of -1, with the exception of 1 and -1. These roots may be written in polar form $\cos\phi^\circ + i\sin\phi^\circ$, where ϕ takes on the following values: $72, 144, 216, 288, 60, 300$. The roots with positive imaginary part have ϕ-values 72, 144, and 60. The product of these roots is $\cos\theta^\circ + i\sin\theta^\circ$, where $\theta = 72 + 144 + 60 = 276$.

Note: This solution illustrates a general method for solving symmetric equations of degree $2n$, by reducing them to equations of degree n. In this example, it is even possible to find non-trigonometric formulas for the roots, by repeated use of the quadratic formula. In particular, the roots of $w^2 + w - 1 = 0$ are $w = \frac{1}{2}(-1 \pm \sqrt{5})$, and the four z-values from the ensuing equation $z + z^{-1} = w$ are fifth roots of 1. They are $z = \frac{1}{4}\left(-1-\sqrt{5} \pm i\sqrt{10-2\sqrt{5}}\right)$ and $z = \frac{1}{4}\left(-1+\sqrt{5} \pm i\sqrt{10+2\sqrt{5}}\right)$. These formulas for fifth roots imply the ruler-and-compass constructibility of a regular pentagon.

OR

Observe that

$$\begin{aligned} z^6 + z^4 + z^3 + z^2 + 1 &= z^6 - z + (z^4 + z^3 + z^2 + z + 1) \\ &= z(z^5 - 1) + \frac{z^5-1}{z-1} \\ &= (z^5-1)\frac{z^2-z+1}{z-1} \\ &= \frac{z^5-1}{z-1}\cdot\frac{z^3+1}{z+1}. \end{aligned}$$

The solution continues as above.

12. (Answer: 058)
Consider the average of all sums of the form

$$|a_1 - a_2| + |a_3 - a_4| + \cdots + |a_{n-1} - a_n|,$$

where n is even and $(a_1, a_2, a_3, \ldots, a_n)$ is a permutation of $(1, 2, 3, \ldots, n)$. Each of the $n!$ sums contains $n/2$ differences of pairs of integers. There are $\binom{n}{2}$ such pairs. For each $k = 1, 2, \ldots, n-1$, there are $n-k$ of these $\binom{n}{2}$ pairs with difference k. Because each of these pairs occurs the same number of times in the $n!$ sums, the average of the differences of all $\frac{n}{2}n!$ pairs is

$$\frac{1}{\binom{n}{2}} \sum_{k=1}^{n-1} k(n-k).$$

Because $k(n-k)$ is the number of subsets $\{a, k+1, b\}$ of $\{1, 2, \ldots, n+1\}$ that have $a < k+1 < b$, it follows that

$$\sum_{k=1}^{n-1} k(n-k) = \binom{n+1}{3}.$$

The average difference is therefore $\frac{\binom{n+1}{3}}{\binom{n}{2}} = \frac{n+1}{3}$. The average sum of $n/2$ differences is $\frac{n(n+1)}{6}$, which equals $55/3$ when $n = 10$. Thus $p + q = 58$.

Note: When $n = 10$, it is easy to calculate the value of $\sum_{k=1}^{n-1} k(n-k)$ directly.

OR

The average is just 5 times the average value of $|a_1 - a_2|$, because the average value of $|a_{2i-1} - a_{2i}|$ is the same for $i = 1, 2, 3, 4, 5$. When $a_1 = k$, the average value of $|a_1 - a_2|$ is

$$\frac{(k-1) + (k-2) + \cdots + 1 + 1 + 2 + \cdots + (10-k)}{9}$$
$$= \frac{1}{9}\left[\frac{k(k-1)}{2} + \frac{(10-k)(11-k)}{2}\right] = \frac{k^2 - 11k + 55}{9}.$$

Thus the average value of the sum is

$$5 \cdot \frac{1}{10} \sum_{k=1}^{10} \frac{k^2 - 11k + 55}{9} = \frac{55}{3},$$

and so $p + q = 58$.

13. (Answer: 065)
Let $AB = c$, $AC = b$, $BC = a$, and notice that $a^2 + b^2 < c^2$. It follows that $\angle C$ is obtuse and that D lies outside $\triangle ABC$. It is given that line AD intersects $\overline{BC}$ at E, the midpoint of $\overline{BC}$. Notice that $\overline{BD}$ is the altitude from B in $\triangle ABE$. Thus

$$\frac{\text{Area}(\triangle ADB)}{\text{Area}(\triangle ABC)} = \frac{\text{Area}(\triangle ADB)}{2\,\text{Area}(\triangle ABE)} = \frac{\frac{1}{2}(BD)(AD)}{2 \cdot \frac{1}{2}(BD)(AE)} = \frac{AD}{2\,AE}. \qquad (1)$$

To find AE, apply the Law of Cosines twice to obtain

$$b^2 = (AE)^2 + \left(\frac{a}{2}\right)^2 - 2(AE)\left(\frac{a}{2}\right)\cos\angle CEA$$

and

$$c^2 = (AE)^2 + \left(\frac{a}{2}\right)^2 - 2(AE)\left(\frac{a}{2}\right)\cos\angle AEB.$$

Now add these two equations, using the fact that $\cos\angle CEA + \cos\angle AEB = 0$, and solve for AE. The result is

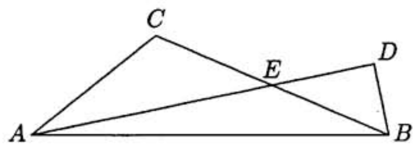
$$AE = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}.$$

Apply the Pythagorean Theorem to find that

$$(AD)^2 + (BD)^2 = c^2$$

and

$$(DE)^2 + (BD)^2 = \tfrac{1}{4}a^2.$$



Substitute $AD = AE + ED$ and subtract to find $(AE)^2 + 2(AE)(ED) = c^2 - \frac{1}{4}a^2$.

Thus

$$\frac{ED}{2\,AE} = \frac{c^2 - \frac{1}{4}a^2 - (AE)^2}{4(AE)^2} = \frac{c^2 - b^2}{2(2b^2 + 2c^2 - a^2)}.$$

Return to (1) to find that

$$\frac{\text{Area}(\triangle ADB)}{\text{Area}(\triangle ABC)} = \frac{AE + ED}{2\,AE} = \frac{1}{2} + \frac{ED}{2\,AE}.$$

When $a = \sqrt{15}$, $b = \sqrt{6}$, and $c = \sqrt{30}$, the desired ratio of areas is $\dfrac{1}{2} + \dfrac{4}{19} = \dfrac{27}{38}$. Hence $m + n = 65$.

14. (Answer: 768)

Let the rectangular solid have width w, length l, and height h, where w, l, and h are positive integers. We will show that the diagonal passes through the interiors of

$$w + l + h - \gcd(w, l) - \gcd(l, h) - \gcd(h, w) + \gcd(w, l, h)$$

of the $1 \times 1 \times 1$ cubes.

Orient the solid in 3-space so that one vertex is at $O = (0, 0, 0)$ and another is at $A = (w, l, h)$. Then $\overline{OA}$ is a diagonal of the solid. Let $P = (x, y, z)$ be a point on this diagonal. Exactly *one* of x, y, z is an integer if and only if P is interior to a face of one of the small cubes. Exactly *two* of x, y, z are integers if and only if P is interior to an edge of one of the small cubes. All *three* of x, y, z are integers if and only if P is a vertex of one of the small cubes. As P moves along the diagonal from O to A, it leaves the interior of a small cube precisely when *at least one* of the coordinates of P is a positive integer. Thus the number of interiors of small cubes through which the diagonal passes is equal to the number of points on the diagonal with at least one positive integer coordinate. Points with positive coordinates on the diagonal $\overline{OA}$ have the form

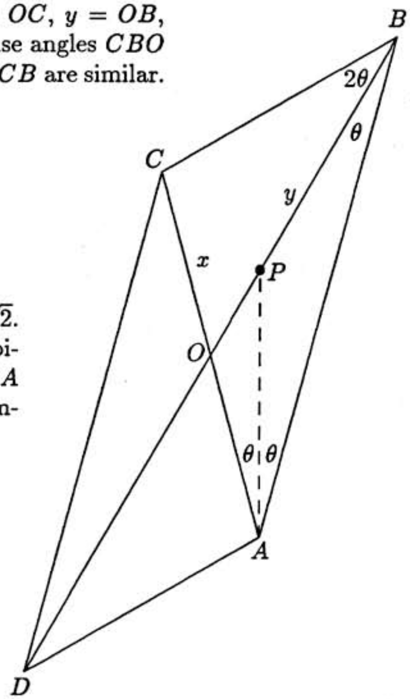
$$P = (wt, lt, ht) \qquad \text{with } 0 < t \leq 1.$$

The first coordinate, wt, will be a positive integer for w values of t, namely for the values $t = 1/w, 2/w, 3/w, \ldots, w/w$. The second coordinate will be an integer for l values of t, and the third coordinate will be an integer for h values of t. The sum $w+l+h$ doubly counts the points with two integer coordinates, however, and it triply counts the points with three integer coordinates. The first two coordinates will be positive integers precisely when t has the form $k/\gcd(w, l)$, for some positive integer k between 1 and $\gcd(w, l)$, inclusive. A similar argument shows that the second and third coordinates will be positive integers for $\gcd(l, h)$ values of t, the third and first coordinates will be positive integers for $\gcd(h, w)$ values of t, and all three will be positive integers for $\gcd(w, l, h)$ values of t. By the inclusion-exclusion principle, P will have one or more positive integer coordinates

$$w + l + h - \gcd(w, l) - \gcd(l, h) - \gcd(h, w) + \gcd(w, l, h)$$

times, which gives 768 when $\{w, l, h\} = \{150, 324, 375\}$.

15. (Answer: 777)



The given data allows us to label $x = OA = OC$, $y = OB$, $\theta = \angle OBA$, and $2\theta = \angle OAB = \angle OBC$. Because angles CBO and CAB are congruent, triangles BCO and ACB are similar. Thus

$$\frac{CB}{CA} = \frac{CO}{CB} = \frac{OB}{BA},$$

or

$$\frac{CB}{2x} = \frac{x}{CB} = \frac{y}{BA}.$$

It follows that $CB = x\sqrt{2}$, and that $BA = y\sqrt{2}$. Now let P be the intersection of $\overline{OB}$ with the bisector of $\angle OAB$. Because angles OAP and OBA are congruent, triangles OPA and OAB are similar. Thus

$$\frac{AP}{BA} = \frac{OA}{OB} = \frac{OP}{OA}.$$

The first equation yields $AP = x\sqrt{2}$. Because $AP = PB$, the second equation yields

$$y^2 - x^2 = xy\sqrt{2}. \tag{1}$$

Apply the Law of Cosines to triangle COB to find that

$$\cos 3\theta = \frac{x^2 + y^2 - (x\sqrt{2})^2}{2xy} = \frac{y^2 - x^2}{2xy},$$

which is $\sqrt{2}/2$, by equation (1). In other words, $3\theta = 45°$, so $\theta = 15°$. It follows that $\angle ACB = 105°$ and $\angle AOB = 135°$, so $r = 105/135 = 7/9 = 0.\overline{7}$ and $1000\,r = 777.\overline{7}$.

OR

Apply the Law of Sines to triangles BOC and ABC to find that

$$BC = \frac{OC \sin 3\theta}{\sin 2\theta} \quad \text{and} \quad BC = \frac{AC \sin 2\theta}{\sin 3\theta},$$

respectively. Because $2 \cdot OC = AC$, it follows that $\sin^2 3\theta = 2\sin^2 2\theta$. Now use the identities $\sin 2\theta = 2\sin\theta\cos\theta$ and $\sin 3\theta = \sin\theta(4\cos^2\theta - 1)$ to produce the equation $(4\cos^2\theta - 1)^2 = 8\cos^2\theta$, then use the identity $2\cos^2\theta = 1 + \cos 2\theta$ to reduce it to $(1 + 2\cos 2\theta)^2 = 4 + 4\cos 2\theta$. This is equivalent to $4\cos^2 2\theta = 3$, hence $\cos 2\theta = \pm\frac{1}{2}\sqrt{3}$. Because it is clear that 2θ is acute, only $\theta = 15°$ is a possibility. Thus $r = 105/135 = 7/9$ and $1000\,r = 777.\overline{7}$.