AIME SOLUTIONS PAMPHLET

FOR STUDENTS AND TEACHERS

3rd ANNUAL
AMERICAN INVITATIONAL
MATHEMATICS EXAMINATION
1985

A prize Examination Sponored by
MATHEMATICAL ASSOCIATION OF AMERICA
SOCIETY OF ACTUARIES
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NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
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This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs. geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

It is hoped that teachers will find the time to share these solutions with their students, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution.

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1. (Answer: 384)

For each $n > 1$, $x_{n-1}x_n = n$. Using this fact for $n = 2, 4, 6$ and 8, one obtains $x_1x_2 \cdots x_8 = (x_1x_2)(x_3x_4)(x_5x_6)(x_7x_8) = 2\cdot4\cdot6\cdot8 = 384$.

Note. Except for the fact that it must be nonzero, the value of x_1 does not affect the solution.

2. (26)

The volume of a cone with a circular base of radius r and height h is given by $\frac{\pi}{3}hr^2$. Denoting the length of the legs of the right triangle by a and b, this implies that

$$\frac{\pi}{3}ba^2 = 800\pi \quad \text{and} \quad \frac{\pi}{3}ab^2 = 1920\pi.$$

Dividing the first of these equations by the second one yields $\frac{a}{b} = \frac{5}{12}$. Hence, $a = \frac{5}{12}b$ and $a^3 = \frac{5}{12}ba^2 = \frac{5}{12}(800)(3) = 1000$. It follows that $a = 10$, $b = 24$ and that the triangle's hypotenuse (by Pythagoras' Theorem) is 26 cm.

3. (198)

First note that one may write c in the form

(1) $$c = a(a^2 - 3b^2) + i[b(3a^2 - b^2) - 107].$$

From this, in view of the fact that c is real, one may conclude that

(2) $$b(3a^2 - b^2) = 107.$$

Since a and b are positive integers, and since 107 is prime, two possible cases arise from (2):

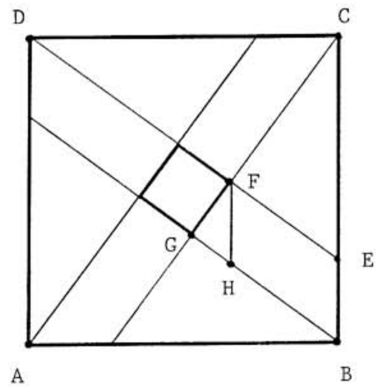
either $b = 107$ and $3a^2 - b^2 = 1$,
or $b = 1$ and $3a^2 - b^2 = 107$.

In the first case, $3a^2 = 107^2 + 1$ would follow. But this is impossible, since $107^2 + 1$ is not a multiple of 3.

In the second case, one finds that $a = 6$ and hence, in view of (1), $c = a(a^2 - 3b^2) = 6(6^2 - 3\cdot1^2) = 198$.

4. (32)

Let A, B, ..., H be the points labeled in the adjoining figure, with FH parallel to EB. Noting that ΔFGH is similar to ΔDCE, it then follows that



$$\frac{FG}{FH} = \frac{DC}{DE},$$

from which

$$FG^2 = \frac{DC^2}{DE^2} \cdot FH^2.$$

In view of the given information, this is equivalent to

$$\frac{1}{1985} = \frac{1}{1 + (1 - \frac{1}{n})^2} \cdot \frac{1}{n^2},$$

from which

$$2n^2 - 2n + 1 = 1985.$$

Since this last equation is equivalent to $2(n - 32)(n + 31) = 0$, and since n is positive, it follows that $n = 32$.

5. (986)

Calculating the first eight terms of the sequence, one finds that it cycles in blocks of six terms; i.e., for $n = 1, 2, 3, \ldots,$ $a_{n+6} = a_n$. More specifically,

$$a_n = \begin{cases} a_1 & \text{if } n = 1, 7, 13, \ldots, \\ a_2 & \text{if } n = 2, 8, 14, \ldots, \\ a_2 - a_1 & \text{if } n = 3, 9, 15, \ldots, \\ -a_1 & \text{if } n = 4, 10, 16, \ldots, \\ -a_2 & \text{if } n = 5, 11, 17, \ldots, \\ a_1 - a_2 & \text{if } n = 6, 12, 18, \ldots . \end{cases}$$

Since the sum of any six consecutive terms of the sequence is zero, if we let s_n be the sum of the first n terms, then

$$s_n = \begin{cases} a_1 & \text{if } n = 1, 7, 13, \ldots, \\ a_1 + a_2 & \text{if } n = 2, 8, 14, \ldots, \\ 2a_2 & \text{if } n = 3, 9, 15, \ldots, 2001, \ldots, \\ 2a_2 - a_1 & \text{if } n = 4, 10, 16, \ldots, 1492, \ldots, \\ a_2 - a_1 & \text{if } n = 5, 11, 17, \ldots, 1985, \ldots, \\ 0 & \text{if } n = 6, 12, 18, \ldots . \end{cases}$$

Therefore,

$$s_{1985} = a_2 - a_1 = 1492$$

and

$$s_{1492} = 2a_2 - a_1 = 1985,$$

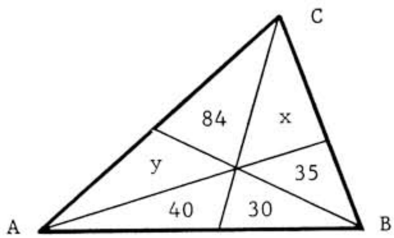
from which $a_2 = 493$ and $s_{2001} = 2a_2 = 986$.

6. (315)

The key to the solution is the fact that if two triangles have the same height, then their areas are proportional to their bases. Therefore, from the figure below, we have the equations

$$\frac{40}{30} = \frac{40 + y + 84}{30 + 35 + x}, \quad \frac{35}{x} = \frac{35 + 30 + 40}{x + 84 + y} \quad \text{and} \quad \frac{84}{y} = \frac{84 + x + 35}{y + 40 + 30}.$$

Solving the first two of these equations simultaneously, one finds that $x = 70$ and $y = 56$. After checking that these values also satisfy the third equation, one may conclude that the area of ΔABC is $30 + 35 + 70 + 84 + 56 + 40$ or 315.



Query. Can you show that such a triangle indeed exists? Can you construct a similar problem with integer areas? For how many of the six small triangles can one choose the area arbitrarily?

7. (757)

Since the prime factorization of positive integers is unique, and since 4 is relatively prime to 5 and 2 is relatively prime to 3, one may conclude that there exist positive integers m and n such that

$$a = m^4, \quad b = m^5, \quad c = n^2 \quad \text{and} \quad d = n^3.$$

Then

$$19 = c - a = n^2 - m^4 = (n - m^2)(n + m^2).$$

Since 19 is a prime, and since $n - m^2 < n + m^2$, it follows that

$$n - m^2 = 1 \quad \text{and} \quad n + m^2 = 19.$$

Therefore, $m = 3$, $n = 10$, $d = 1000$, $b = 243$ and $d - b = 1000 - 243 = 757$.

8. (61)

Two preliminary observations are needed:

(i) Each A_i should be 2 or 3, because for any scheme meeting this condition, $M < 1$; while for any other choice of the A_i's, $M > 1$.

(ii) There is only one way to sum seven integers, each of them 2 or 3, and to obtain 19 : two of them must be 2, while the other five must be 3.

In view of the above, and to make M as small as possible, one must round down (to 2) the two numbers with smallest decimal parts (i.e., a_1 and a_2), and round up (to 3) the other five a_i's. Thereby one finds that

$$M = \max \{ |a_2 - 2|, \ |a_3 - 3| \} = \max \{ .61, \ .35 \} = .61,$$

and that $100M = 61$.

Note. More generally, one can show that if any a_i's are listed so that their decimal parts are in increasing order, if the sum of the a_i's is D, and if the sum of their integral parts is I, then the last $D - I$ of the a_i's should be rounded up (to the next integer), while the others should be rounded down.

Such unusual methods of rounding are necessary, for example, in the apportionment of congressional seats to the individual states in the United States. (The U.S. Constitution requires each state to be represented in proportion to its fraction of the U.S. population, but each state must also have an integer number of representatives, and the total size of the House is fixed.) For informative discussions and other solutions to the apportionment problem the reader is advised to read the following article in The American Mathematical Monthly: M.L. Balinski and H.P. Young, The Quota Method of Apportionment, vol.82(1975), pp 701-730.

9. (49)

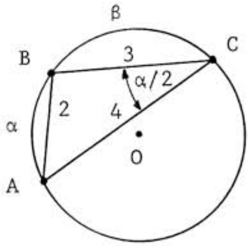
Since any two chords of equal length subtend equal angles, the parallelism of the chords is irrelevant. Therefore, we may choose points A, B and C, as shown in the first accompanying figure, so that $AB = 2$, $BC = 3$ and (since $\widehat{AC} = \alpha + \beta$) $AC = 4$. Since $\angle ACB = \alpha/2$, by the Law of Cosines we have

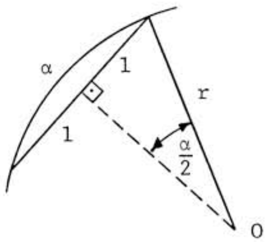
$$\cos\frac{\alpha}{2} = \frac{AC^2 + BC^2 - AB^2}{2\cdot AC\cdot BC} = \frac{4^2 + 3^2 - 2^2}{2\cdot 4\cdot 3} = \frac{7}{8}.$$

Therefore, one finds that

$$\cos\alpha = 2\cos^2\frac{\alpha}{2} - 1 = 2\cdot\frac{49}{64} - 1 = \frac{17}{32},$$

and that the desired answer is $17 + 32$ or 49.





Alternate Solution. Attacking the problem more directly, let r denote the radius of the circle, and note that $\sin\frac{\alpha}{2} = \frac{1}{r}$, as shown in the second figure above. One similarly finds that $\sin\frac{\beta}{2} = \frac{3}{2r}$ and $\sin\frac{\alpha+\beta}{2} = \frac{2}{r}$. Since $\sin(\frac{\alpha}{2} + \frac{\beta}{2}) = \sin\frac{\alpha}{2}\cos\frac{\beta}{2} + \cos\frac{\alpha}{2}\sin\frac{\beta}{2}$, it follows that

$$(1) \qquad \frac{2}{r} = \frac{1}{r}\sqrt{1 - \frac{9}{4r^2}} + \frac{3}{2r}\sqrt{1 - \frac{1}{r^2}}.$$

Solving this for $\frac{1}{r^2}$, one finds that it is equal to $\frac{15}{64}$. Upon checking that this root is not extraneous to (1), it follows that

$$\cos\alpha = 1 - 2\sin^2\frac{\alpha}{2} = 1 - 2\cdot\frac{15}{64} = \frac{17}{32},$$

as in the first solution.

10. (600)

Introduce the notation

(1) $$f(x) = \lfloor 2x \rfloor + \lfloor 4x \rfloor + \lfloor 6x \rfloor + \lfloor 8x \rfloor,$$

and observe that if n is a positive integer, then from (1)

(2) $$f(x+n) = f(x) + 20n$$

follows. In particular, this means that if an integer k can be expressed in the form $f(x_o)$ for some real number x_o, then for $n = 1, 2, 3, \ldots$ one can express $k + 20n$ similarly; i.e., $k + 20n = f(x_o) + 20n = f(x_o + n)$. In view of this, one may restrict attention to determining which of the first 20 positive integers are generated by $f(x)$ as x ranges through the half-open interval $(0, 1]$.

Next observe that as x increases, the value of $f(x)$ changes only when either $2x$, $4x$, $6x$ or $8x$ attains an integral value, and that the change in $f(x)$ is always to a new, higher value. In the interval $(0, 1]$ such changes occur precisely when x is of the form m/n, where $1 \le m \le n$ and $n = 2, 4, 6$ or 8. There are 12 such fractions; in increasing order they are:

$$\frac{1}{8}, \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{3}{8}, \frac{1}{2}, \frac{5}{8}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8} \text{ and } 1.$$

Therefore, only 12 of the first 20 positive integers can be represented in the desired form. Since $1000 = (50)(20)$, in view of (2), this implies that in each of the 50 sequences,

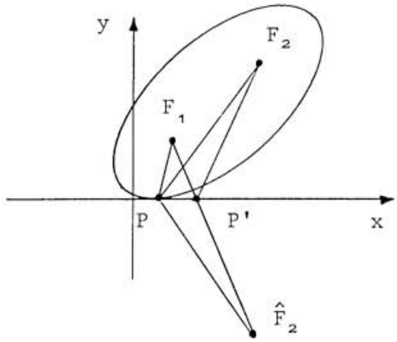
$$1, 2, 3, \ldots, 20;\ 21, 22, 23, \ldots, 40;\ \ldots\ ;\ 981, 982, 983, \ldots, 1000,$$

of 20 consecutive integers only 12 can be so expressed, leading to a total of $(50)(12)$ or 600 positive integers of the desired form.

11. (85)

Let F_1 and F_2 denote the given foci, $(9, 20)$ and $(49, 55)$, respectively, and let $\hat{F}_2$: $(49, -55)$ be the reflection of F_2 in the x-axis. We claim that k, the length of the major axis of the ellipse, is equal to the length of the segment $F_1\hat{F}_2$, which is easily found to be 85 by the distance formula between two points in the plane.

To prove our claim, let P be the point of tangency of the ellipse to the x-axis, and assume that the segment $F_1\hat{F}_2$ intersects the x-axis at the point P', distinct from P, as shown in the adjacent figure, where the location of the foci is purposefully distorted. Then, by the Triangle Inequality, one finds that



(1) $$PF_1 + P\hat{F}_2 > F_1\hat{F}_2 = P'F_1 + P'\hat{F}_2 = P'F_1 + P'F_2.$$

However, by the definition of the ellipse, for every point Q on the ellipse, $QF_1 + QF_2 = k$. Hence, in particular,

(2) $$PF_1 + PF_2 = k.$$

Moreover, for every point Q outside the ellipse, one must have $QF_1 + QF_2 > k$. Since, by the definition of tangency, P' must lie outside the ellipse, in particular it follows that

(3) $$P'F_1 + P'F_2 > k.$$

From (2) and (3) one may therefore conclude that

$$PF_1 + P\hat{F}_2 = PF_1 + PF_2 < P'F_1 + P'F_2,$$

which is contrary to (1). This contradiction establishes the falsity of the assumption that P and P' are distinct, and hence completes the proof of the claim.

12. (182)

For $n = 0, 1, 2, \ldots$, let a_n be the probability that the bug is at vertex A after crawling exactly n meters. Then

(1) $$a_{n+1} = \frac{1}{3}(1 - a_n),$$

because the bug can be at vertex A after crawling $n+1$ meters if and only if

(i) it was not at A following a crawl of n meters (this has probability $1 - a_n$)

and (ii) from one of the other vertices it heads toward A (this has probability $\frac{1}{3}$).

Now since $a_0 = 1$ (i.e., the bug starts at vertex A), from (1) we have

$$a_1 = 0,\quad a_2 = \frac{1}{3},\quad a_3 = \frac{2}{9},\quad a_4 = \frac{7}{27},\quad a_5 = \frac{20}{81},\quad a_6 = \frac{61}{243},$$

and $p = a_7 = \frac{182}{729}$, leading to 182 as the answer to the problem.

Note. The above calculations may be somewhat simplified by observing that for $n \geq 1$ the numerators of a_n and a_{n+1} sum to 3^{n-1}. Alternately, one may write (1) in the form

$$a_n - \frac{1}{4} = -\frac{1}{3}(a_{n-1} - \frac{1}{4}),$$

which (upon successive substitutions, starting with $a_0 - \frac{1}{4} = \frac{3}{4}$) yields the explicit formula

$$a_n = \frac{1}{4} + (-\frac{1}{3})^n(\frac{3}{4}).$$

13. (401)

More generally, we will show that if a is a positive integer and if d_n is the greatest common divisor of $a+n^2$ and $a+(n+1)^2$, then the maximum value of d_n is $4a+1$, attained when $n = 2a$. It will follow that for the sequence under consideration the answer is $4(100)+1$ or 401.

To prove the above, note that if d_n divides $a+(n+1)^2$ and $a+n^2$, then it also divides their difference; i.e.,

(1) $$d_n \mid (2n+1).$$

Now, since $2(a+n^2) = n(2n+1) + (2a-n)$, it follows from (1) that

(2) $$d_n \mid (2a-n).$$

Hence from (1) and (2), $d_n \mid ((2n+1) + 2(2a-n))$, or

(3) $$d_n \mid (4a+1).$$

Consequently, $1 \leq d_n \leq 4a+1$, so $4a+1$ is the largest possible value of d_n. It is attained, since for $n = 2a$ we have

$$a + n^2 = a + (2a)^2 = a(4a+1)$$

and

$$a + (n+1)^2 = a + (2a+1)^2 = (a+1)(4a+1).$$

14. (25)

Assume that a total of n players participated in the tournament. We will obtain two expressions in n: one by considering the total number of points gathered by all of the players, and one by considering the number of points gathered by the losers (10 lowest scoring contestants) and those gathered by the winners (other $n - 10$ contestants) separately. To obtain the desired expressions, we will use the fact that if k players played against one another, then they played a total of $k(k-1)/2$ games, resulting in a total of $k(k-1)/2$ points to be shared among them.

In view of the last observation, the n players gathered a total of $n(n-1)/2$ points in the tournament. Similarly, the losers had $10 \cdot 9/2$ or 45 points in games among themselves; since this accounts for half of their points, they must have had a total of 90 points. In games among themselves the $n - 10$ winners similarly gathered $(n-10)(n-11)/2$ points; this also accounts for only half of their total number of points (the other half coming from games against the losers), so their total was $(n-10)(n-11)$ points. Thus we have the equation

$$n(n-1)/2 = 90 + (n-10)(n-11),$$

which is equivalent to

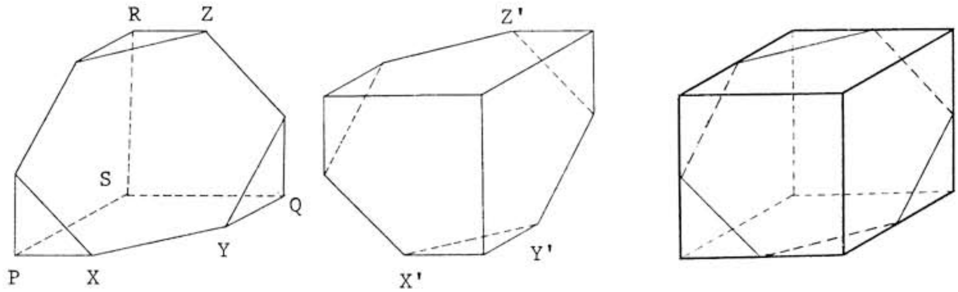
$$n^2 - 41n + 400 = 0.$$

Since the left member of this equation may be factored as $(n-16)(n-25)$, it follows that $n = 16$ or 25. We discard the first of these in view of the following observation: if there were only 16 players in the tournament, then there would have been only 6 winners, and the total of their points would have been 30 points, resulting in an average of 5 points for each of them. This is less than the 90/10 or 9 points gathered, on the average, by each of the losers! Therefore, $n = 25$; i.e., there were 25 players in the tournament.

Note. The AIME participants should recognize that there exists at least one tournament with 25 contestants meeting the conditions of the problem, for otherwise the problem would not have been posed. They are urged to attempt the reconstruction of such a tournament.

15. (864)

At each of the vertices P, Q, R and S, marked in the first figure below, each of the three facial angles measures 90°. Consequently, the polyhedron may be immediately recognized as a portion of a cube. Moreover, two such polyhedra may be fitted together along their hexagonal faces. To accomplish this, flip and rotate by 60° the second polyhedron (as shown in the second figure below), and then slide them together until X and X', Y and Y', as well as Z and Z' coincide. As shown in the third figure below, the result of the procedure is a cube of volume 12^3 or 1728 cm^3. Therefore, the volume of the polyhedron under consideration is 1728/2 or 864 cm^3.



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Readers of this Pamphlet are encouraged to submit alternate (and perhaps more elegant) solutions, insightful remarks, interesting extensions, answers to the queries, as well as other related materials to the Chairman of the AIME for possible inclusion in the journal ARBELOS. Constructive criticism of the AIME problems and solutions will also be most appreciated.