AMERICAN MATHEMATICS COMPETITIONS

8th ANNUAL
AMERICAN INVITATIONAL MATHEMATICS EXAMINATION
(AIME)
TUESDAY, MARCH 20, 1990

Sponsored by

Mathematical Association of America
Society of Actuaries   Mu Alpha Theta
National Council of Teachers of Mathematics
Casualty Actuarial Society   American Statistical Association
American Mathematical Association of Two-Year Colleges
American Mathematical Society

1. DO NOT OPEN THIS BOOKLET UNTIL TOLD TO DO SO BY YOUR PROCTOR.
2. This is a 15-question, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers; i.e., there is neither partial credit nor a penalty for wrong answers.
3. Scratch paper, graph paper, ruler, compass, protractor and eraser are permitted. *Calculators and slide rules are not permitted.*
4. Please print the following:

Name: ____
Last   First   Middle initial

Home address: ____
Street Address

____
City   State or Province   Zip or Postcode

______________________   ______   ______
Home Phone including Area Code   Sex (M or F)   Your age

______________________   ______
Full Name of School   Grade Level (e.g.,11)

5. My score on the 1990 AHSME was ☐   I took the 1990 AHSME on (date): ______________
6. A combination of the AIME and AHSME scores is used to determine eligibility for participation in the U. S. A. Mathematical Olympiad (USAMO) to be given on April 24, 1990. Please check one box:

   If I qualify for the USAMO, I agree to take it.   YES ☐   NO ☐

   (*Your school must also agree to administer the USAMO before you can take it.*)
7. Record all your answers, and certain other information, on the AIME answer form. Your Examination Manager will instruct you how to complete the form. Only the answer form and this cover will be collected from you.

1. The increasing sequence $2, 3, 5, 6, 7, 10, 11, \ldots$ consists of all positive integers that are neither the square nor the cube of a positive integer. Find the 500^{th} term of this sequence.

2. Find the value of $\left(52+6\sqrt{43}\right)^{3/2}-\left(52-6\sqrt{43}\right)^{3/2}$.

3. Let P_1 be a regular r-gon and P_2 be a regular s-gon $(r \geq s \geq 3)$ such that each interior angle of P_1 is $\frac{59}{58}$ as large as each interior angle of P_2. What is the largest possible value of s?

4. Find the positive solution to

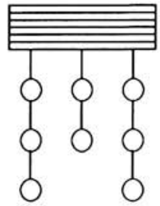
$$\frac{1}{x^2-10x-29}+\frac{1}{x^2-10x-45}-\frac{2}{x^2-10x-69}=0.$$

5. Let n be the smallest positive integer that is a multiple of 75 and has exactly 75 positive integral divisors, including 1 and itself. Find $n/75$.

6. A biologist wants to calculate the number of fish in a lake. On May 1 she catches a random sample of 60 fish, tags them, and releases them. On September 1 she catches a random sample of 70 fish and finds that 3 of them are tagged. To calculate the number of fish in the lake on May 1, she assumes that 25% of these fish are no longer in the lake on September 1 (because of death and emigrations), that 40% of the fish present on September 1 were not in the lake on May 1 (because of births and immigrations) and that the numbers of untagged fish and tagged fish in the September 1 sample are representative of the total population. What does the biologist calculate for the number of fish in the lake on May 1?

7. A triangle has vertices $P=(-8,5)$, $Q=(-15,-19)$ and $R=(1,-7)$. The equation of the bisector of $\angle P$ can be written in the form $ax+2y+c=0$. Find $a+c$.

8. In a shooting match, eight clay targets are arranged in two hanging columns of three each and one column of two, as pictured. A marksman is to break all eight targets according to the following rules: (1) The marksman first chooses a column from which a target is to be broken. (2) The marksman must then break the lowest remaining unbroken target in the chosen column. If these rules are followed, in how many different orders can the eight targets be broken?



9. A fair coin is to be tossed ten times. Let i/j, in lowest terms, be the probability that heads never occur on consecutive tosses. Find $i + j$.

10. The sets $A = \{z : z^{18} = 1\}$ and $B = \{w : w^{48} = 1\}$ are both sets of complex roots of unity. The set $C = \{zw : z \in A \text{ and } w \in B\}$ is also a set of complex roots of unity. How many distinct elements are in C?

11. Someone observed that $6! = 8 \cdot 9 \cdot 10$. Find the largest positive integer n for which $n!$ can be expressed as the product of $n-3$ consecutive positive integers.

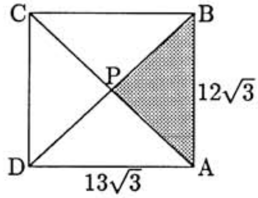
12. A regular 12-gon is inscribed in a circle of radius 12. The sum of the lengths of all sides and diagonals of the 12-gon can be written in the form

$$a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6},$$

where a, b, c, and d are positive integers. Find $a + b + c + d$.

13. Let $T = \{9^k : k \text{ is an integer}, 0 \le k \le 4000\}$. Given that 9^{4000} has 3817 digits and that its first (leftmost) digit is 9, how many elements of T have 9 as their leftmost digit?

14. The rectangle $ABCD$ at the right has dimensions $AB = 12\sqrt{3}$ and $BC = 13\sqrt{3}$. Diagonals $\overline{AC}$ and $\overline{BD}$ intersect at P. If triangle ABP is cut out and removed, edges $\overline{AP}$ and $\overline{BP}$ are joined, and the figure is then creased along segments $\overline{CP}$ and $\overline{DP}$, we obtain a triangular pyramid, all four of whose faces are isosceles triangles. Find the volume of this pyramid.



15. Find $ax^5 + by^5$ if the real numbers a, b, x and y satisfy the equations

$$ax + by = 3, \qquad ax^2 + by^2 = 7, \qquad ax^3 + by^3 = 16, \qquad ax^4 + by^4 = 42.$$

SOLUTIONS

A 1990 Solutions Pamphlet will be sent to exam managers within a few weeks.

WRITE TO US!

Questions and comments about the problems and solutions for this AIME (but **not** requests for the Solutions Pamphlet) should be addressed to:

Prof Elgin H Johnston, AIME Chairman
Department of Mathematics
Iowa State University, Ames, IA 50011 USA

Comments about administrative arrangements and orders for any publications listed below should be addressed to:

Prof Walter E Mientka, AMC Executive Director
Department of Mathematics and Statistics
University of Nebraska, Lincoln, NE 68588-0322 USA

1990 USAMO

The USA Mathematical Olympiad is a 5-question, $3\frac{1}{2}$ hour, essay-type examination. Top-scoring AHSME/AIME students will be invited to take the USAMO on April 24, 1990. (See the AHSME or AIME Teachers' Manual for more details.) The best way to prepare for the USAMO is to study the exams from previous years and to review the contents of the ARBELOS. Copies may be ordered as indicated below.

PUBLICATIONS

MINIMUM ORDER: $5 (before handling fee), US FUNDS ONLY. Canada and US orders must be prepaid. Orders are mailed 4th class, unless you specify 1st class, in which case add $3.00 or 20% of total order, whichever is larger, with a maximum of $10.00. Make checks payable to the American Mathematics Competitions.

FOREIGN ORDERS: do **NOT** prepay; an invoice will be sent.

Examinations: Each price is for **one** copy of an exam **and** its solutions for one year. Specify the years you want and how many copies of each. All prices effective to July 1, 1990.

- **AJHSME** (Junior High Exam), 1985-1988, 50 cents per copy per year.
- **AHSME** 1972-90, 50 cents per copy per year.
- **AIME** 1983-89, $2 per copy per year.
- USA and International Mathematical **Olympiads** (together), 1976-89, $2 per copy per year.
- National **Summary** of Results and Awards, 1980-89, $4 per copy per year.

Books (Exams and solutions):

- Contest Problem Book I, AHSMEs 1950-60, $8.50.
- Contest Problem Book II, AHSMEs 1961-65, $8.50.
- Contest Problem Book III, AHSMEs 1966-72, $10.00.
- Contest Problem Book IV, AHSMEs 1973-82, $11.00.
- USA Mathematical Olympiads, 1972-86, $13.00.
- International Mathematical Olympiads, 1959-77, $10.00.
- International Mathematical Olympiads, 1978-85, $11.00.

Journal

- The ARBELOS (short articles and challenging problems); recommended especially for AIME and USAMO qualifiers. Six volumes, 1982-1987, $7.00 each.