AMERICAN MATHEMATICS COMPETITIONS

AIME SOLUTIONS PAMPHLET
FOR STUDENTS AND TEACHERS

7th ANNUAL
AMERICAN INVITATIONAL MATHEMATICS EXAMINATION
(AIME)
TUESDAY, MARCH 21, 1989

Sponsored by

Mathematical Association of America
Society of Actuaries Mu Alpha Theta
National Council of Teachers of Mathematics
Casualty Actuarial Society American Statistical Association
American Mathematical Association of Two-Year Colleges
American Mathematical Society

This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs. geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

It is hoped that teachers will find the time to share these solutions with their students, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution.

Questions and comments about the problems and solutions (but **not** requests for the Solutions Pamphlet) should be addressed to:

Professor Elgin H Johnston, AIME Chairman
Department of Mathematics
Iowa State University
Ames, IA 50011 USA

Order prior year Examination questions and Solutions Pamphlets or Problem Books from:

Professor Walter E Mientka, CAMC Executive Director
Department of Mathematics and Statistics
University of Nebraska
Lincoln, NE 68588-0322 USA

1. (Answer: 869)
The data

$$\begin{array}{rcccccc} 4{\cdot}3{\cdot}2{\cdot}1 & + & 1 & = & 5^2 & = & (3\cdot 2-1)^2 \\ 5{\cdot}4{\cdot}3{\cdot}2 & + & 1 & = & 11^2 & = & (4\cdot 3-1)^2 \\ 6{\cdot}5{\cdot}4{\cdot}3 & + & 1 & = & 19^2 & = & (5\cdot 4-1)^2 \end{array}$$

suggest that $(k+1)(k)(k-1)(k-2)+1=[k(k-1)-1]^2=[(k^2-k)-1]^2$. The calculations

$$\begin{aligned}(k+1)(k)(k-1)(k-2)+1 &= \big[(k+1)(k-2)\big]\big[k(k-1)\big]+1\\ &= (k^2-k-2)(k^2-k)+1\\ &= (k^2-k)^2-2(k^2-k)+1\\ &= \big[(k^2-k)-1\big]^2\end{aligned}$$

show that this is true. Thus $\sqrt{(31)(30)(29)(28)+1}=30^2-30-1=869$.

2. (Answer: 968)
For $3\le k\le 10$, each choice of k points will yield a convex polygon with k vertices. Because k points can be chosen from 10 in $\binom{10}{k}$ ways, the answer to the problem is

$$\begin{aligned}&\binom{10}{3}+\binom{10}{4}+\cdots+\binom{10}{10}\\ &\qquad=\left[\binom{10}{0}+\binom{10}{1}+\cdots+\binom{10}{10}\right]-\left[\binom{10}{0}+\binom{10}{1}+\binom{10}{2}\right]\\ &\qquad=(1+1)^{10}-(1+10+45)\\ &\qquad=968.\end{aligned}$$

Query. Where have we used the stipulation that the polygons are convex?

3. (Answer: 750)
Since $\frac{n}{810}=0.\mathrm{d}25\mathrm{d}25\mathrm{d}25\ldots$, we have $1000\frac{n}{810}=\mathrm{d}25.\mathrm{d}25\mathrm{d}25\ldots$. Subtracting gives

$$\frac{999}{810}n=1000\frac{n}{810}-\frac{n}{810}=\mathrm{d}25=100\mathrm{d}+25.$$

Consequently $999n=810(100\mathrm{d}+25)$, which leads to $37n=750(4\mathrm{d}+1)$. Noting that 750 and 37 are relatively prime, we see that $4\mathrm{d}+1$ must be a multiple of 37. Since d is a single digit, $\mathrm{d}=9$ and hence $n=750$.

4. (Answer: 675)
Since a, b, c, d and e are consecutive integers, $b+c+d=3c$ and $a+b+c+d+e=5c$. Let m and n be positive integers such that $b+c+d=m^2$ and $a+b+c+d+e=n^3$. Then

$$3c=m^2 \tag{1}$$

and

$$5c=n^3 \tag{2}$$

From (1) we see that $3|m$ and hence $3|c$. (If j and k are integers, $j|k$ is read "j divides k" and means that j is a factor of k.) Therefore, (2) implies that $3|n$ and hence that $3^3|c$. From (2) we also find that $5|n$, which leads to $5^2|c$. Consequently, $25\cdot 27$ divides c. It is easy to verify that $c=25\cdot 27=675$ is the solution we seek.

5. (Answer: 283)
Let r be the probability of getting heads when the coin is tossed once. Then the probability of getting k heads out of n tosses is $\binom{n}{k}r^k(1-r)^{n-k}$. Consequently, the information given in the problem leads to the equation

$$\binom{5}{1}r(1-r)^4=\binom{5}{2}r^2(1-r)^3,$$

from which $r=0, \frac{1}{3}$ or 1. Since the desired probability is not 0, we may conclude that $r=\frac{1}{3}$. Thus the probability of obtaining exactly 3 heads in 5 tosses is

$$\frac{i}{j}=\binom{5}{3}r^3(1-r)^2=\binom{5}{3}\left(\tfrac{1}{3}\right)^3\left(1-\tfrac{1}{3}\right)^2=\frac{40}{243}$$

and $i+j=40+243=283$.

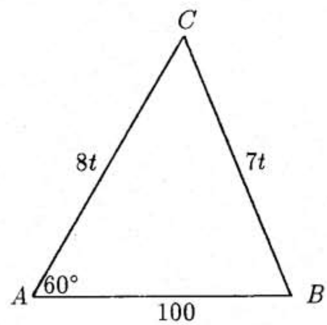
6. (Answer: 160)
Suppose that after t seconds, Allie and Billie meet at point C. Then $\triangle ABC$ has sides of $AB=100$, $AC=8t$ and $BC=7t$, with $\overline{BC}$ opposite the 60° angle. Hence, by the Law of Cosines,

$$(7t)^2=100^2+(8t)^2-2(100)(8t)\cos 60^\circ,$$

from which

$$0=3t^2-160t+2000=(3t-100)(t-20).$$

Thus $t=20$ or $t=\dfrac{100}{3}$. Since the meeting takes place at the earliest possible time, we must have $t=20$ and hence $AC=8\cdot 20=160$ meters.



7. (Answer: 925)
We seek the values of k, n and d such that

$$36 + k = (n - d)^2 \quad (1)$$
$$300 + k = n^2 \quad (2)$$
$$596 + k = (n + d)^2. \quad (3)$$

Subtracting (1) from (3), we find that $4nd = 560$, from which

$$nd = 140. \quad (4)$$

Multiplying equation (2) by 2 and subtracting the result from the sum of (1) and (3), we find that $2d^2 = 32$, giving $d = \pm 4$. Combining this with (4), we find $n = \pm 35$. Using (2) again, we have $k = n^2 - 300 = 35^2 - 300 = 925$.

Note. One may also find k by solving the equation

$$\sqrt{36 + k} + \sqrt{596 + k} = 2\sqrt{300 + k}.$$

8. (Answer: 334)
Observe that subtracting 3 times the second equation from the sum of the first equation and 3 times the third equation gives an equation with the desired expression on the left. Thus

$$16x_1 + 25x_2 + 36x_3 + 49x_4 + 64x_5 + 81x_6 + 100x_7 = 1(1) - 3(12) + 3(123) = 334.$$

More formally, we can deduce the relation mentioned above by finding constants a, b and c such that

$$an^2 + b(n + 1)^2 + c(n + 2)^2 = (n + 3)^2 \quad (1)$$

holds for all n. Considering (1) as a polynomial identity in n, we expand and simplify both sides, then equate coefficients of like powers of n. We obtain the equations

$$\begin{aligned} a + b + c &= 1 \\ 2b + 4c &= 6 \\ b + 4c &= 9. \end{aligned}$$

The solution of this system is $a = 1$, $b = -3$ and $c = 3$.

Note. For the sake of completeness, one should check that the system of three equations given in the problem does have a solution. One such solution is $x_1 = 797/4$, $x_2 = -916/4$, $x_3 = 319/4$ and $x_4 = x_5 = x_6 = x_7 = 0$.

It is interesting to note that the number of variables and their values are of little significance. The reader may wish to investigate generalizations of these results to problems in which the coefficients are cubes, fourth powers, etc.

9. (Answer: 144)
It is clear that $n \geq 134$. We can get an upper bound on n by noting that

$$\begin{aligned} n^5 &= 133^5 + 110^5 + 84^5 + 27^5 \\ &< 133^5 + 110^5 + (27+84)^5 \\ &< 3(133)^5 \\ &< \frac{3125}{1024}(133)^5 \\ &= \left(\frac{5}{4}\right)^5 (133)^5. \end{aligned}$$

Thus $n < (\frac{5}{4})(133)$, giving $n \leq 166$. Next note that, when an integer is raised to the fifth power, its units digit is unchanged. It follows that n has the same units digit as the sum $133 + 110 + 84 + 27$; i.e., the units digit of n is 4, and n is one of the four numbers 134, 144, 154, 164. Since $133 \equiv 1 \pmod 3$, $110 \equiv 2 \pmod 3$, $84 \equiv 0 \pmod 3$ and $27 \equiv 0 \pmod 3$, we have

$$n^5 = 133^5 + 110^5 + 84^5 + 27^5 \equiv 1^5 + 2^5 \equiv 0 \pmod 3.$$

This means that n is a multiple of 3, and we conclude that $n = 144$.

Note. Euler's original conjecture was, that for any integer $n \geq 3$, the equation

$$x_1^n + x_2^n + x_3^n + \cdots + x_{n-1}^n = x_n^n$$

has no non-trivial integer solutions. The "spoilers of Euler" were L. J. Lander, T. R. Parkin and J. L. Selfridge. Their work was published in *Mathematics of Computation*, **21**(1967), 446–459. Recently, N. D. Elkies, (a USAMO winner) showed that Euler's conjecture is also false in the case $n = 4$.

10. (Answer: 994)

First note that

$$\frac{\cot\gamma}{\cot\alpha+\cot\beta}=\frac{\dfrac{\cos\gamma}{\sin\gamma}}{\dfrac{\cos\alpha\sin\beta+\sin\alpha\cos\beta}{\sin\alpha\sin\beta}}=\frac{\cos\gamma\sin\alpha\sin\beta}{\sin\gamma\sin(\alpha+\beta)}=\frac{\cos\gamma\sin\alpha\sin\beta}{\sin^2\gamma},$$

and that

$$\frac{\sin\alpha}{a}=\frac{\sin\beta}{b}=\frac{\sin\gamma}{c}$$

implies

$$\frac{\sin\alpha\sin\beta}{\sin^2\gamma}=\frac{ab}{c^2}.$$

Thus

$$\frac{\cot\gamma}{\cot\alpha+\cot\beta}=\frac{ab\cos\gamma}{c^2}.$$

Hence, by the Law of Cosines,

$$\frac{\cot\gamma}{\cot\alpha+\cot\beta}=\frac{a^2+b^2-c^2}{2c^2}=\frac{1989c^2-c^2}{2c^2}=994.$$

11. (Answer: 947)

Let M be the mode and m be the mean. We may assume, without loss of generality, that $M\geq m$. For D to be as large as possible, $M=1000$, since if $M=1000-k$, then increasing M by k increases m by no more than k. As a result, D certainly does not decrease.

Given that $M=1000$, we must make m as small as possible. Now 1000 must occur in the sample at least twice, for otherwise it could not be the unique mode. If 1000 occurs exactly twice, then every other number in the sample must occur once. In this case, m will be smallest if the other $121-2$ values are 1, 2, 3, … , 119. This leads to a mean of

$$\frac{\frac{(119)(119+1)}{2}+2000}{121}=\frac{9140}{121}.$$

If $M=1000$ occurs exactly 3 times, then every other value can occur at most twice, and m will be smallest if the other 118 sample values are 1, 1, 2, 2, 3, 3, … , 59, 59. We then have a mean of

$$\frac{\frac{(118)(59+1)}{2}+3000}{121}=\frac{6540}{121}.$$

If the mode occurs 4 times, the smallest possible mean is

$$\frac{\frac{(117)(39+1)}{2}+4000}{121}=\frac{6340}{121}.$$

If the mode occurs 5 times, the smallest mean is

$$\frac{\frac{(116)(29+1)}{2}+5000}{121}=\frac{6740}{121}$$

and if the mode occurs 6 times, the smallest mean is

$$\frac{\frac{(115)(23+1)}{2}+6000}{121}=\frac{7380}{121}.$$

Finally, if $M=1000$ occurs exactly n times, with $n\geq 7$, then

$$m\geq \frac{1000n}{121}\geq \frac{7000}{121}.$$

The above shows that the smallest m occurs when $M=1000$ occurs exactly 4 times, and this m is $\frac{6340}{121}=52+\frac{48}{121}$. Thus the largest value of D is $1000-(52+\frac{48}{121})$, and $\lfloor D\rfloor=947$.

Query. Can you state and solve an analogous problem for real number data?

12. (Answer: 137)

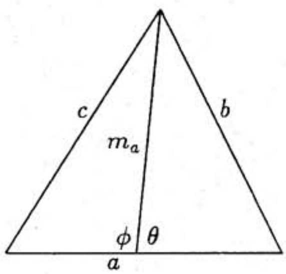
We first show that if a, b and c are the three sides of a triangle and m_a is the median to side a, then

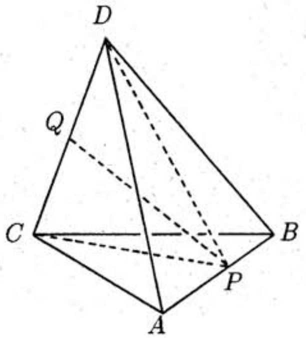
$$m_a^2=\frac{1}{4}(2b^2+2c^2-a^2). \tag{1}$$

We then apply (1) three times in order to find the desired d^2.

To prove (1), consider the figure shown below. Apply the Law of Cosines to each of the two smaller triangles to get

$$\begin{aligned} m_a^2 &= \frac{1}{2}(m_a^2+m_a^2)\\ &= \frac{1}{2}[(c^2-\frac{1}{4}a^2+am_a\cos\phi)\\ &\qquad +(b^2-\frac{1}{4}a^2+am_a\cos\theta)]\\ &= \frac{1}{4}(2b^2+2c^2-a^2)\\ &\qquad +\frac{1}{2}am_a(\cos\phi+\cos(\pi-\phi))\\ &= \frac{1}{4}(2b^2+2c^2-a^2). \end{aligned}$$





Next, in the tetrahedron shown on the left, let P be the midpoint of $\overline{AB}$ and Q be the midpoint of $\overline{CD}$. We apply (1) to find $(PC)^2$ (since $\overline{PC}$ is a median of ΔABC) and $(PD)^2$ (since $\overline{PD}$ is a median of ΔABD). We find

$$(PC)^2 = \frac{1}{4}[2(AC)^2 + 2(BC)^2 - (AB)^2] = \frac{1009}{4}$$

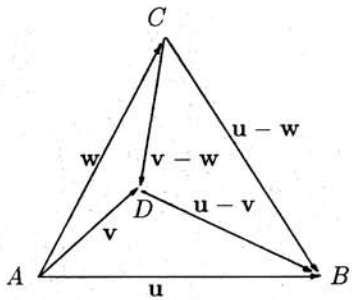
and

$$(PD)^2 = \frac{1}{4}[2(AD)^2 + 2(BD)^2 - (AB)^2] = \frac{425}{4}.$$

Finally, we again use (1) to find $d^2 = (PQ)^2$ from ΔCDP:

$$(PQ)^2 = \frac{1}{4}[2(PC)^2 + 2(PD)^2 - (CD)^2] = 137.$$

Alternate Solution. Introduce the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ to denote the directed edges from A. (See the accompanying figure.) The other three edges, with orientations as indicated, are given by $\mathbf{u}-\mathbf{v}$, $\mathbf{u}-\mathbf{w}$ and $\mathbf{v}-\mathbf{w}$. Moreover, the vector from A to the midpoint of $\overline{AB}$ is $\frac{1}{2}\mathbf{u}$ and the vector from A to the midpoint of $\overline{CD}$ is $\frac{1}{2}(\mathbf{v}+\mathbf{w})$. Consequently, the vector from the midpoint of $\overline{CD}$ to the midpoint of $\overline{AB}$ is $\frac{1}{2}\mathbf{u} - \frac{1}{2}(\mathbf{v}+\mathbf{w})$. We seek the square of the length of this last vector. Recalling that for a vector $\mathbf{x}$, $|\mathbf{x}|^2 = \mathbf{x}\cdot\mathbf{x}$, we have



$$d^2 = \frac{1}{4}(\mathbf{u}-\mathbf{v}-\mathbf{w})\cdot(\mathbf{u}-\mathbf{v}-\mathbf{w}) = \frac{1}{4}(|\mathbf{u}|^2 + |\mathbf{v}|^2 + |\mathbf{w}|^2 - 2\mathbf{u}\cdot\mathbf{v} - 2\mathbf{u}\cdot\mathbf{w} + 2\mathbf{v}\cdot\mathbf{w}). \quad (1)$$

To find $\mathbf{u}\cdot\mathbf{v}$, $\mathbf{u}\cdot\mathbf{w}$ and $\mathbf{v}\cdot\mathbf{w}$, note that $|\mathbf{x}-\mathbf{y}|^2 = |\mathbf{x}|^2 + |\mathbf{y}|^2 - 2\mathbf{x}\cdot\mathbf{y}$, which implies

$$2\mathbf{x}\cdot\mathbf{y} = |\mathbf{x}|^2 + |\mathbf{y}|^2 - |\mathbf{x}-\mathbf{y}|^2. \quad (2)$$

Applying (2) to (1) yields the general formula

$$d^2 = \frac{1}{4}(|\mathbf{v}|^2 + |\mathbf{w}|^2 + |\mathbf{u}-\mathbf{v}|^2 + |\mathbf{u}-\mathbf{w}|^2 - |\mathbf{u}|^2 - |\mathbf{v}-\mathbf{w}|^2). \quad (3)$$

The given measurements yield $d^2 = 137$.

Note. For an alternate derivation of (3) the reader may wish to consult Straszewicz's excellent *Mathematical Problems and Puzzles from the Polish Olympiad*, published by Pergamon Press in 1965, but unfortunately long out of print.

13. (Answer: 905)

We first show that, given any set of 11 consecutive integers from $\{1, 2, 3, \ldots, 1989\}$, at most 5 of these 11 can be elements of S. We prove this fact for the set $T = \{1, 2, 3, \ldots, 11\}$, but the same proof works for any set of 11 consecutive integers. Consider the following partition of T, where each subset was formed so that it can contribute at most one element to S:

$$\{1,5\} \qquad \{2,9\} \qquad \{3,7\} \qquad \{4,11\} \qquad \{6,10\} \qquad \{8\}. \tag{1}$$

If it were possible to have 6 elements of T in S, then each of the sets in (1) would have to contribute exactly one element. That this is impossible is shown by the following chain of implications:

$$8 \in S \Longrightarrow 1 \notin S \Longrightarrow 5 \in S \Longrightarrow 9 \notin S \Longrightarrow 2 \in S \Longrightarrow 6 \notin S \Longrightarrow 10 \in S$$
$$\Longrightarrow 3 \notin S \Longrightarrow 7 \in S \Longrightarrow 11 \notin S \Longrightarrow 4 \in S \Longrightarrow 8 \notin S.$$

With the aid of (1), or otherwise, it is easy to find a 5-element subset of T that satisfies the key property of S (i.e., no two numbers differ by 4 or 7). One such set is

$$T' = \{1, 3, 4, 6, 9\}.$$

We also find (perhaps to our surprise) that T' has the remarkable property of allowing for a periodic continuation. That is, if I denotes the set of integers, then

$$S' = \{k + 11n \mid k \in T' \quad \text{and} \quad n \in I\}$$

also has the property that no two elements in the set differ by 4 or 7. Moreover, since $1989 = 180 \cdot 11 + 9$, it is clear that S cannot have more than $181 \cdot 5 = 905$ elements. Because the largest element in T' is 9, it follows that the set

$$S = S' \cap \{1, 2, 3, \ldots, 1989\}$$

has 905 elements and hence shows that the upper bound of 905 on the size of the desired set can be attained. This completes the argument.

Note. The reader may wish to find other 5-element subsets of $\{1, 2, 3, \ldots, 11\}$ that exhibit the key property of S. Which of these subsets can be used, as above, to generate a maximal S?

The reader is also encouraged to explore similar problems with other pairs (triples, etc.) of integers in place of 4 and 7, and to find the appropriate motivations for the choice of 11 as the size of the blocks of integers considered in the above solution.

14. (Answer: 490)

If

$$\begin{aligned} k = (a_3a_2a_1a_0)_{-3+i} &= a_3(-3+i)^3 + a_2(-3+i)^2 + a_1(-3+i) + a_0 \\ &= a_3(-18+26i) + a_2(8-6i) + a_1(-3+i) + a_0 \\ &= (-18a_3 + 8a_2 - 3a_1 + a_0) + (26a_3 - 6a_2 + a_1)i \end{aligned}$$

is a real integer, then its imaginary part must vanish. Thus

$$26a_3 - 6a_2 + a_1 = 0. \tag{1}$$

Since $a_3,\ a_2,\ a_1 \in \{0,\ 1,\ \ldots,\ 9\}$ and $a_3 \neq 0$, we see that (1) can hold only if $a_3 = 1$ or $a_3 = 2$.

If $a_3 = 1$, then $6a_2 - a_1 = 26$ and the restrictions on a_2 and a_1 force $a_2 = 5$ and $a_1 = 4$. In this case, we have

$$k = (a_3a_2a_1a_0)_{-3+i} = -18a_3 + 8a_2 - 3a_1 + a_0 = 10 + a_0.$$

Since $a_0 \in \{0,\ 1,\ 2,\ \ldots,\ 9\}$, we see that k can be any one of 10, 11, ..., 19.

If $a_3 = 2$, then $6a_2 - a_1 = 52$, leading to $a_2 = 9$ and $a_1 = 2$. It follows that $k = 30 + a_0$ and k can be any one of 30, 31, ..., 39. Adding the possibilities from the two cases gives the answer

$$(10 + 11 + \cdots + 19) + (30 + 31 + \cdots + 39) = 490.$$

Note. For more about complex bases, the reader is referred to W. Gilbert's article in the March 1984 issue of *Mathematics Magazine*.

15. (Answer: 108)

We are given length information about the three segments through P. Our strategy is to translate one of these segments to form a new triangle (inside of $\triangle ABC$) for which we know all three sides, and hence the area. We then multiply this area by an appropriate ratio to obtain the area of $\triangle ABC$.

We first find the lengths of $\overline{CP}$ and $\overline{PF}$. To this end, observe that

$$\frac{\text{Area}(\triangle BPC)}{\text{Area}(\triangle BAC)} = \frac{PD}{AD} = \frac{6}{6+6} = \frac{1}{2} \tag{1}$$

and

$$\frac{\text{Area}(\triangle APC)}{\text{Area}(\triangle ABC)} = \frac{PE}{BE} = \frac{3}{3+9} = \frac{1}{4}. \tag{2}$$

Thus

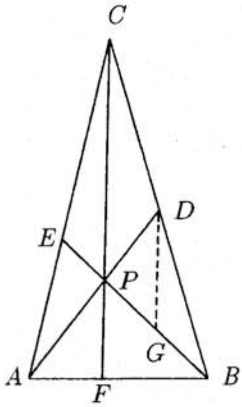
$$\frac{PF}{CF} = \frac{\text{Area}(\triangle APB)}{\text{Area}(\triangle ACB)}$$
$$= \frac{\text{Area}(\triangle ACB) - \text{Area}(\triangle APC) - \text{Area}(\triangle BPC)}{\text{Area}(\triangle ACB)} \qquad (3)$$
$$= 1 - \frac{1}{4} - \frac{1}{2} = \frac{1}{4}.$$

Since $CF = 20$, it follows that $PF = 5$ and $CP = 15$. Furthermore,

$$\frac{BD}{CD} = \frac{\text{Area}(\triangle ADB)}{\text{Area}(\triangle ACD)} = \frac{\text{Area}(\triangle PDB)}{\text{Area}(\triangle PDC)}$$
$$= \frac{\text{Area}(\triangle ADB) - \text{Area}(\triangle PDB)}{\text{Area}(\triangle ACD) - \text{Area}(\triangle PDC)} = \frac{\text{Area}(\triangle APB)}{\text{Area}(\triangle ACP)} = 1,$$

where the last equality results from dividing equation (3) by equation (2).

Next construct $\overline{DG}$ parallel to $\overline{CF}$, with G on $\overline{PB}$ as shown. We show that $\triangle GDP$ is a right triangle with sides of $\frac{9}{2}$, 6 and $\frac{15}{2}$ and then show that the area of $\triangle GDP$ is $\frac{1}{8}$ of the area of $\triangle ABC$. It will then follow that the desired answer is $8 \cdot \frac{27}{2} = 108$.



To establish the above claims, first note that $\triangle BDG \sim \triangle BCP$. Since $BD = \frac{1}{2}BC$, the sides of these two triangles are in a ratio of 1:2. It follows that $DG = \frac{1}{2}PC = \frac{15}{2}$ and $PG = GB = \frac{1}{2}PB = \frac{9}{2}$. Since $PD = 6$ was given, we see that $\triangle GDP$ is a right triangle as claimed. Next note that

$$\frac{\text{Area}(\triangle PBC)}{\text{Area}(\triangle GBD)} = \left(\frac{2}{1}\right)^2 = 4 \qquad \text{and} \qquad \frac{\text{Area}(\triangle GBD)}{\text{Area}(\triangle GPD)} = \frac{BG}{PG} = 1.$$

Using these ratios with (1) gives

$$\text{Area}(\triangle ABC) = \frac{\text{Area}(\triangle ABC)}{\text{Area}(\triangle PBC)} \cdot \frac{\text{Area}(\triangle PBC)}{\text{Area}(\triangle GBD)} \cdot \frac{\text{Area}(\triangle GBD)}{\text{Area}(\triangle GPD)} \cdot \text{Area}(\triangle GPD)$$
$$= 2 \cdot 4 \cdot 1 \cdot \frac{27}{2} = 108.$$

Note. Implicit in this solution is a method for constructing $\triangle ABC$ with straightedge and compass from the given data. The reader is invited to explore the conditions such data must satisfy in order to ensure that $\triangle ABC$ exists (and is unique).

Alternate Solution. Let $P = (0,0)$, $D = (6,0)$, $A = (-6,0)$, $E = (h,k)$ and $B = (-3h,-3k)$. Solving the equations for $\overline{AE}$ and $\overline{BD}$ simultaneously, we find $C = (3h+12, 3k)$. Next, the coordinates of F can be found by solving the equations of $\overline{CP}$ and $\overline{AB}$ simultaneously; the result is $F = (-4-h, -k)$. Finally, solving the equations $h^2 + k^2 = 9$ and $(4+h)^2 + k^2 = 25$ (arising from $PE = 3$ and $CF = 20$, respectively) one finds that $h = 0$ and $k = 3$. Once we have the coordinates of A, B and C, we can find that the area of $\triangle ABC$ is 108.