AMERICAN MATHEMATICS COMPETITIONS

18th ANNUAL

AMERICAN INVITATIONAL MATHEMATICS EXAMINATION

(AIME)

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SOLUTIONS PAMPHLET

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This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution. *Remember that reproduction of these solutions is prohibited by copyright.*

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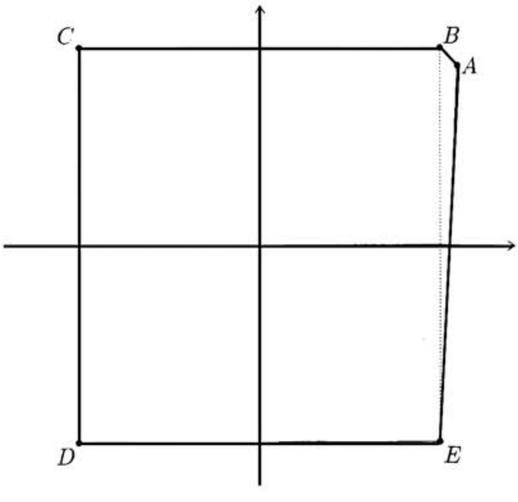
1. (Answer: 008)
The number 10^n can be expressed as the product of 2^n and 5^n, neither of which contains the digit 0 for $n \leq 7$. Since $5^8 = 380625$, and all other pairs of positive integers whose product is 10^8 contain at least one trailing 0, the requested value of n is 8.

2. (Answer: 021)
It follows from the problem statement that the coordinates of A, B, C, D, and E are $A = (u, v), B = (v, u), C = (-v, u), D = (-v, -u)$, and $E = (v, -u)$. The pentagon can be partitioned into rectangle $BCDE$ and triangle ABE, whose areas are $2u \cdot 2v = 4uv$ and $\frac{1}{2}2u(u - v) = u^2 - uv$, so

$$\text{Area}(ABCDE) = 4uv + u^2 - uv = u(3v + u).$$

Thus $u(3v + u) = 451 = 11 \cdot 41$. Because $0 < v < u$, it follows that $u = 11$, $3v + 11 = 41$, $v = 10$, and $u + v = 21$.

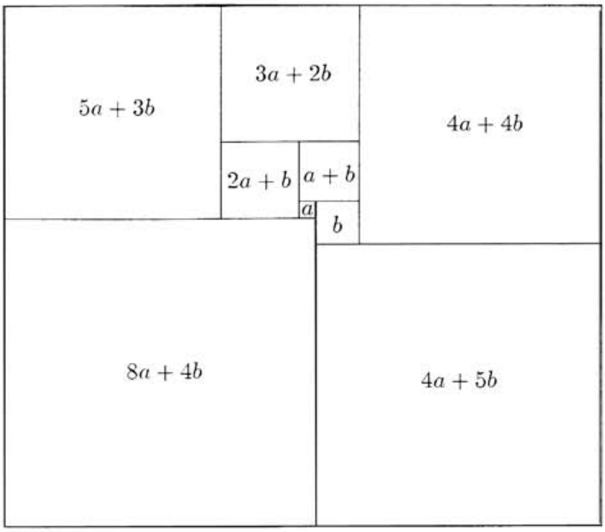


3. (Answer: 667)
The problem statement implies that

$$\frac{2000 \cdot 1999}{2 \cdot 1} a^2 b^{1998} = \frac{2000 \cdot 1999 \cdot 1998}{3 \cdot 2 \cdot 1} a^3 b^{1997}.$$

This is equivalent to $b = 666a$. Because a and b have no common divisors greater than 1, it follows that $a = 1$, $b = 666$, and $a + b = 667$.

4. (Answer: 260)
Let a and b be the lengths of the sides of the two smallest squares, with $a < b$. In the diagram below, each square has been labeled with the length of its sides.



The length $13a + 7b$ of the left side of the rectangle is equal to the length $8a + 9b$ of the right side, so $5a = 2b$. This implies that there is a positive number t for which $a = 2t$ and $b = 5t$. Thus the width-to-height ratio for the rectangle is $\dfrac{12a+9b}{8a+9b} = \dfrac{69t}{61t} = \dfrac{69}{61}$. Because the dimensions of the rectangle are relatively prime positive integers, they must be 69 and 61. Therefore the perimeter is $2(69 + 61) = 260$.

5. (Answer: 026)
Suppose that the first box has k_1 marbles, b_1 of which are black, and that the second box has k_2 marbles, b_2 of which are black. Without loss of generality, assume that $k_1 < k_2$. Thus $k_1 + k_2 = 25$ and $(b_1/k_1) \cdot (b_2/k_2) = 27/50$, so $50b_1b_2 = 27k_1k_2$. The latter equation implies that 5 divides either k_1 or k_2. Because $k_1 + k_2 = 25$, both k_1 and k_2 must be divisible by 5. One possibility is that $k_1 = 5$ and $k_2 = 20$, in which case $b_1b_2 = 54$, so $b_1 = 3$ and $b_2 = 18$, because $b_1 < k_1$ and $b_2 < k_2$. In this case, the probability of obtaining two white marbles is $(2/5) \cdot (2/20) = 1/25$. The only other possibility is that $k_1 = 10$ and $k_2 = 15$, in which case $b_1b_2 = 81$, so b_1 and b_2 must each be 9. The probability of obtaining two white marbles is $(1/10) \cdot (6/15) = 1/25$ in this case too. Hence $m + n = 26$.

6. (Answer: 997)
From
$$\frac{x+y}{2} = 2 + \sqrt{xy},$$
it follows that
$$x + y - 2\sqrt{xy} = 4,$$
$$(\sqrt{y} - \sqrt{x}\,)^2 = 4, \text{ and}$$
$$\sqrt{y} - \sqrt{x} = 2.$$
Because $y = (2 + \sqrt{x}\,)^2 = x + 4 + 4\sqrt{x}$ is an integer, it follows that $4\sqrt{x}$ must be an integer. Consequently $16x$ is a perfect square, and $\sqrt{x}$ is an integer. From $(2+\sqrt{x}\,)^2 < 10^6$, it follows that $\sqrt{x} < 998$. Thus the 997 solutions are $(x, y) = \left(n^2, (n+2)^2\right)$, for $n = 1, 2, \ldots, 997$.

7. (Answer: 005)
Notice that
$$\begin{aligned} 5 \cdot 29 \cdot \frac{m}{n} &= \left(x + \frac{1}{z}\right)\left(y + \frac{1}{x}\right)\left(z + \frac{1}{y}\right) \\ &= xyz + x + \frac{1}{z} + y + \frac{1}{x} + z + \frac{1}{y} + \frac{1}{xyz} \\ &= 1 + 5 + 29 + \frac{m}{n} + 1. \end{aligned}$$
Thus $144 \cdot \frac{m}{n} = 36$, so that $\frac{m}{n} = \frac{1}{4}$ and $m + n = 5$.

OR

Because
$$5 = x + \frac{1}{z} = x + xy = x + x\left(29 - \frac{1}{x}\right) = 30x - 1,$$
it follows that $x = \frac{1}{5}$, $y = 24$, and $z = \frac{5}{24}$. Thus $z + \frac{1}{y} = \frac{5}{24} + \frac{1}{24} = \frac{1}{4}$.

8. (Answer: 052)
When the cone is held point down, the liquid in the container forms a cone that is similar to the container, the ratio of similarity being $\frac{3}{4}$. Thus the volume of the liquid is $\left(\frac{3}{4}\right)^3$ times the volume of the container. When the cone is held point up, the air in the container forms a cone whose height is h and whose volume is $1 - \left(\frac{3}{4}\right)^3 = \frac{37}{64}$ times the volume of the container. Because the cone of air is similar to the container, $\frac{37}{4^3} = \frac{h^3}{12^3}$, so $h^3 = 3^3 \cdot 37$. It follows that the depth of the liquid is $12 - h = 12 - 3\sqrt[3]{37}$. Thus $m + n + p = 12 + 3 + 37 = 52$.

9. (Answer: 025)

Let $u = \log_{10} x$, $v = \log_{10} y$, and $w = \log_{10} z$. The given equations can be rewritten as

$$\begin{aligned} uv - u - v + 1 &= \log_{10} 2 \\ vw - v - w + 1 &= \log_{10} 2 \\ wu - w - u + 1 &= 1, \end{aligned}$$

and then as

$$\begin{aligned} (u-1)(v-1) &= \log_{10} 2 \\ (v-1)(w-1) &= \log_{10} 2 \\ (w-1)(u-1) &= 1. \end{aligned}$$

It follows from the first two equations that $u = w$, and the third equation then implies that $u = w = 2$ or $u = w = 0$. In the first case, $v = \log_{10} 20$, so $x_1 = 100$, $y_1 = 20$, and $z_1 = 100$. In the second case, $v = \log_{10} 5$, so $x_2 = 1$, $y_2 = 5$, and $z_2 = 1$. Thus $y_1 + y_2 = 25$.

10. (Answer: 173)

Let $S = x_1 + x_2 + x_3 + \cdots + x_{100}$, so $x_k = (S - x_k) - k$ for all integers k between 1 and 100, inclusive. Thus $k + 2x_k = S$ for all such k. Summing these equations for k=1, 2, 3, ..., 100 yields

$$\frac{100 \cdot 101}{2} + 2S = 100S,$$

from which $S = \dfrac{2525}{49}$ follows. Thus $x_{50} = \dfrac{S - 50}{2} = \dfrac{75}{98}$, and $m + n = 173$.

11. (Answer: 248)

Because $1000 = 2^3 5^3$, each a/b may be written in the form $2^m 5^n$, where $-3 \le m \le 3$ and $-3 \le n \le 3$. It follows that each a/b appears exactly once in the expansion of

$$\left(2^{-3} + 2^{-2} + \cdots + 2^2 + 2^3\right)\left(5^{-3} + 5^{-2} + \cdots + 5^2 + 5^3\right).$$

Thus $S = \dfrac{2^4 - 2^{-3}}{2-1} \cdot \dfrac{5^4 - 5^{-3}}{5-1} = \dfrac{127}{8} \cdot \dfrac{19531}{125} = 2480 + \dfrac{437}{1000}$, so $\dfrac{S}{10} = 248 + \dfrac{437}{10000}$.

12. (Answer: 177)

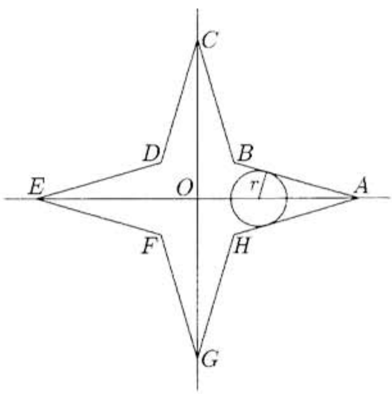
From the given identities $f(x) = f(398 - x)$, $f(x) = f(2158 - x)$, and $f(x) = f(3214 - x)$, derive the following identities:

$$\begin{aligned}
f(x) &= f(2158 - x) = f(3214 - (2158 - x)) = f(1056 + x),\\
f(x) &= f(1056 + x) = f(2158 - (1056 + x)) = f(1102 - x),\\
f(x) &= f(1056 + x) = f(1102 - (1056 + x)) = f(46 - x), \text{ and}\\
f(x) &= f(46 - x) = f(398 - (46 - x)) = f(352 + x).
\end{aligned}$$

It follows from the last identity that f is periodic, and that the period of f divides 352. Thus every value in the list is found among $f(0)$, $f(1)$, $\ldots$, $f(351)$. The identity $f(x) = f(398 - x)$ implies that $f(200)$, $f(201)$, $\ldots$, $f(351)$ are found among $f(0)$, $f(1)$, $\ldots$, $f(199)$, and the identity $f(x) = f(46 - x)$ implies that $f(0)$, $f(1)$, $\ldots$, $f(22)$ are found among $f(23)$, $f(24)$, $\ldots$, $f(199)$. Thus there can be at most 177 different values in the list. To see that the values $f(23)$, $f(24)$, $\ldots$, $f(199)$ can be distinct, consider the function $f(x) = \cos\left(\frac{360}{352}(x - 23)\right)$, whose argument is interpreted in degrees. It is routine to verify the required identities $f(x) = f(398 - x)$, $f(x) = f(2158 - x)$, and $f(x) = f(3214 - x)$, and to see that $1 = f(23) > f(24) > f(25) > \cdots > f(199) = -1$.

13. (Answer: 731)

Set up a coordinate system in which the axes coincide with the highways. The points that can be reached within six minutes lie on or within circles of radius $14\left(\frac{1}{10} - t\right) = \frac{7}{5} - 14t$ centered at points $(\pm 50t, 0)$ or $(0, \pm 50t)$ on the axes, where $0 \le t \le \frac{1}{10}$. The radius r of each circle is thus linearly related to c, the distance from its center to the origin $O = (0,0)$: namely $r = \frac{7}{5} - \frac{7}{25}c$, or $r = \frac{7}{25}(5 - c)$. Given any one of the circles centered on the positive x-axis, $\frac{r}{5-c} = \frac{7}{25}$ is the sine of the angle formed by the x-axis and a tangent drawn from $A = (5, 0)$. It follows that these circles share two common external tangent lines, and that the region in question is a nonconvex octagon $ABCDEFGH$, where B is in the first quadrant. Notice that B is on the line $y = x$, which is one of the four axes of symmetry of the region. An equation for the common external tangent $\overline{AB}$ is $7x + 24y = 35$, because $\tan \angle OAB = \frac{7}{24}$. Set $x = y$ to find that $B = \left(\frac{35}{31}, \frac{35}{31}\right)$. The area of $ABCDEFGH$ is 8 times the area of OAB, or $8 \cdot \dfrac{1}{2} \cdot 5 \cdot \dfrac{35}{31} = \dfrac{700}{31}$. Thus $m + n = 731$.

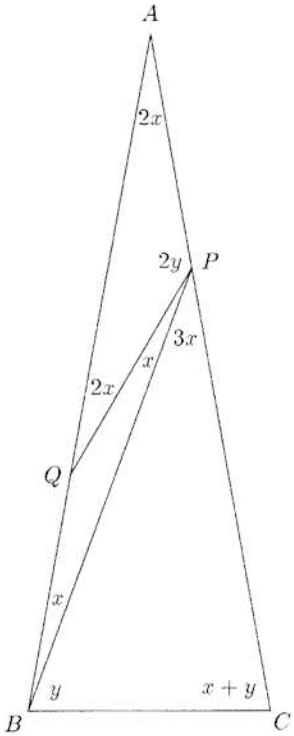


14. (Answer: 571)

Let x denote the degree measure of angle QPB. Because angle AQP is exterior to isosceles triangle BQP, its measure is $2x$, and angle PAQ has the same measure. Because angle BPC is exterior to triangle BPA, its measure is $3x$. Let y denote the measure of angle PBC. It follows that the measure of angle ACB is $x + y$, and that $4x + 2y = 180$. Two of the angles of triangle APQ have measure $2x$, and thus the measure of angle APQ is $2y$. It follows that $AQ = 2 \cdot AP \cdot \sin y$. Because $AB = AC$ and $AP = QB$, it also follows that $AQ = PC$. Now apply the Law of Sines to triangle PBC to find that

$$\frac{\sin 3x}{BC} = \frac{\sin y}{PC} = \frac{\sin y}{2 \cdot AP \cdot \sin y} = \frac{1}{2 \cdot BC},$$

because $AP = BC$. Hence $\sin 3x = \frac{1}{2}$. This and $4x < 180$ imply that $3x = 30$ and $x = 10$. Thus $y = 70$ and $r = \dfrac{10 + 70}{2 \cdot 70} = \dfrac{4}{7}$, so $1000r = 571 + \dfrac{3}{7}$.



OR

Let $u = \angle ACB = \angle ABC$. Then $\angle A = \angle AQP = 180 - 2u$, $\angle APQ = 4u - 180$, $\angle QBP = \angle QPB = 90 - u$, $\angle BQC = \angle BCQ = 90 - \frac{1}{2}u$, $\angle CQP = \frac{5}{2}u - 90$, and $\angle QCP = \frac{3}{2}u - 90$. Apply the Law of Sines to triangles APQ and CPQ to obtain

$$\frac{\sin(4u - 180)}{\sin(180 - 2u)} = \frac{AQ}{AP} = \frac{PC}{PQ} = \frac{\sin(\frac{5}{2}u - 90)}{\sin(\frac{3}{2}u - 90)}.$$

This is equivalent to

$$\frac{-\sin 4u}{\sin 2u} = \frac{\cos \frac{5}{2}u}{\cos \frac{3}{2}u},$$

or $-2\cos 2u \cos \frac{3}{2}u = \cos \frac{5}{2}u$. Use the identity $2\cos\alpha\cos\beta = \cos(\alpha - \beta) + \cos(\alpha + \beta)$ to obtain $\cos \frac{7}{2}u + \cos \frac{5}{2}u + \cos \frac{1}{2}u = 0$, and then again to obtain $2\cos 3u \cos \frac{1}{2}u + \cos \frac{1}{2}u = 0$. This implies that $\cos 3u = -\frac{1}{2}$, so that $u = 40$ or $u = 80$. Because $4u$ must be greater than 180, it follows that $u = 80$. Thus $\angle APQ = 4u - 180 = 140$ and $r = 4/7$, as above.

15. (Answer: 927)

Run the process backwards. Start by picking up the card labeled 2000. Next pick up the card labeled 1999, place it on the top of the stack, and bring the bottom card to the top of the stack. Next pick up the card labeled 1998, place it on top of the stack, and bring the bottom card to the top of the stack. The card labeled 1999 is now at the top of a three-card stack. Notice that the top card of an m-card stack will become the top card of a $2m$-card stack after m more cards have been picked up (and m cards have been moved from the bottom of the stack to the top). It follows by induction that the card labeled 1999 is the top card when the number of cards in the stack is $3 \cdot 2^k$ for any nonnegative integer k that satisfies $3 \cdot 2^k < 2000$. In particular, the last time that this happens is just after $3 \cdot 2^9 = 1536$ cards have been picked up. The cards remaining on the table are labeled 1 through 464. After each of the cards labeled 464, 463, ..., 2 is picked up and placed on top of the stack, another card is brought from the bottom of the stack to the top. Finally, the card labeled 1 is placed on top of the stack and the stack is in its original state. This puts $2 \cdot 463 + 1 = 927$ cards on top of the card labeled 1999.

OR

Because the process causes the cards on the table to appear in ascending order, the card labeled 1999 is the next-to-last card placed on the table. To keep track of that card, first notice that, when a stack of 2^m cards is dealt in this way, the next-to-last card placed on the table begins at position 2^{m-1} in the stack; then apply the process to a stack of $2^{11} = 2048$ cards. After 48 of the cards have been placed on the table and 48 more cards have been moved from the top of the stack to the bottom, a 2000-card stack remains. Remove the cards that are on the table. The next-to-last card that will be placed on the table from the 2000-card stack is the card that began at position 1024 in the 2048-card stack. The position of that card in the 2000-card stack is $1024 - (48 + 48) = 928$, so the number of cards above it is 927.