AMERICAN MATHEMATICS COMPETITIONS

AIME SOLUTIONS PAMPHLET
FOR STUDENTS AND TEACHERS

13th ANNUAL
AMERICAN INVITATIONAL MATHEMATICS EXAMINATION
(AIME)
THURSDAY, March 23, 1995

Sponsored by

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This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs. geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution. *Remember that reproduction of these solutions is prohibited by copyright.*

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1. (Answer: 255)
Observe that the area of S_{i+1} is one fourth that of S_i, and that three fourths of S_{i+1} is not inside S_i. Therefore the area enclosed by at least one of S_1, S_2, S_3, S_4, S_5 is

$$1+\frac{3}{4}\left(\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}\right)=\frac{1279}{1024}.$$

Hence $m-n=1279-1024=255$.

2. (Answer: 025)
Taking logarithms of both sides of the equation, we find that

$$\log\left(\sqrt{1995}x^{\log x}\right)=\log x^2,$$

where all the logarithms are to the base 1995. From this we obtain

$$\frac{1}{2}\log 1995+(\log x)(\log x)=2\log x,$$

which leads to

$$(\log x)^2-2\log x+\frac{1}{2}=0.$$

Solving this last equation gives

$$\log x=1\pm\sqrt{\frac{1}{2}}.$$

Since these values for $\log x$ are both real, the original equation has two positive roots; call them r_1 and r_2. Since $\log r_1r_2=\log r_1+\log r_2=2$, the product of these roots is

$$r_1r_2=1995^2=(2000-5)^2=2000^2-10\cdot 2000+25.$$

The last three digits of this number are 025.

3. (Answer: 067)
Since the net movement must be two steps right (R) and two steps up (U), there must be at least four steps. The point $(2,2)$ can be reached in exactly four steps if the sequence is some permutation of R, R, U, U. These four steps can be permuted in

$$\frac{4!}{2!2!}=6$$

ways. Each of these sequences has probability $(1/4)^4$ of occurring. Thus the probability of reaching $(2,2)$ in exactly 4 steps is $6/4^4$.

In moving to $(2,2)$, the total number of steps must be even, since an odd number of steps would reach a lattice point with one even coordinate and one odd coordinate. Next consider the possibility of reaching $(2,2)$ in six steps. A six-step sequence must include the steps R, R, U, U in some order, as well as a pair consisting of R, L (left) or U, D (down), in some order. The steps R, R, U, U, U, D can be permuted in

$$\frac{6!}{3!2!1!} = 60$$

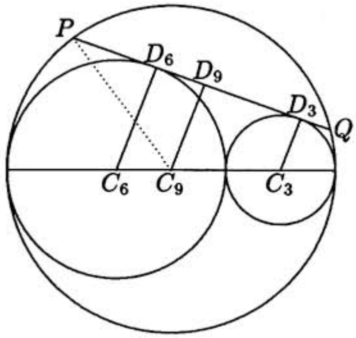
ways, but for 12 of these sequences — namely those that start with some permutation of R, R, U, U — the object actually reaches $(2,2)$ in four steps. A similar analysis holds for the steps R, R, U, U, R, L. Thus there are $2(60-12) = 96$ six-step sequences that reach $(2,2)$, but that do not do so until the sixth step. Each of these 96 sequences occurs with probability $1/4^6$.

Considering the four- and six-step possibilities, we find that the probability of reaching $(2,2)$ in six or fewer steps is

$$\frac{6}{4^4} + \frac{96}{4^6} = \frac{3}{64}.$$

Thus $m + n = 67$.

4. (Answer: 224)
Let $\overline{PQ}$ be the tangent chord, and let C_3, C_6, and C_9 be the centers of the circles of radii 3, 6, and 9, respectively. We see that $C_6C_9 = 3$ and $C_9C_3 = 6$. Let D_3, D_6, and D_9, respectively, be the feet of the perpendiculars from C_3, C_6, and C_9 to $\overline{PQ}$. Then D_3 and D_6 are points of tangency, and $\overline{C_3D_3}$, $\overline{C_6D_6}$, and $\overline{C_9D_9}$ are parallel. It follows that



$$C_9D_9 = \frac{1 \cdot C_3D_3 + 2 \cdot C_6D_6}{3} = 5.$$

Now apply the Pythagorean Theorem to right triangle C_9PD_9 to find that

$$(PQ)^2 = 4(D_9P)^2 = 4\left[(C_9P)^2 - (C_9D_9)^2\right] = 4(9^2 - 5^2) = 224.$$

5. (Answer: 051)
Let the roots be r_1, r_2, r_3, r_4, where $r_1r_2 = 13 + i$ and $r_3 + r_4 = 3 + 4i$. Because the polynomial has real coefficients and none of the roots is real, the roots occur in conjugate pairs, say $r_3 = \overline{r_1}$ and $r_4 = \overline{r_2}$. It follows that $r_3r_4 = \overline{r_1r_2} = 13 - i$ and $r_1 + r_2 = \overline{r_3 + r_4} = 3 - 4i$. The polynomial is therefore

$$[x^2 - (3 - 4i)x + (13 + i)][x^2 - (3 + 4i)x + (13 - i)]$$

$$= x^4 - 6x^3 + 51x^2 - 70x + 170.$$

In particular, $b = 51 = (13 + i) + (3 - 4i)(3 + 4i) + (13 - i)$.

Query. Because the roots occur in conjugate pairs, the given polynomial can be factored as the product of two quadratics that have *real* coefficients. What are these factors?

6. (Answer: 589)
Let $n = p^r q^s$, where p and q are distinct primes. Then $n^2 = p^{2r} q^{2s}$, so n^2 has

$$(2r + 1)(2s + 1)$$

factors. For each factor less than n, there is a corresponding factor greater than n. By excluding the factor n, we see that there must be

$$\frac{(2r + 1)(2s + 1) - 1}{2} = 2rs + r + s$$

factors of n^2 that are less than n. Because n has $(r + 1)(s + 1)$ factors (including n itself), and because every factor of n is also a factor of n^2, there are

$$2rs + r + s - [(r + 1)(s + 1) - 1] = rs$$

factors of n^2 that are less than n but not factors of n. When $r = 31$ and $s = 19$, there are $rs = 589$ such factors.

7. (Answer: 027)

Let $s = \sin t + \cos t$ and $p = \sin t \cos t$. It is given that $1 + s + p = \frac{5}{4}$; thus $p = \frac{1}{4} - s$. It then follows that

$$1 = \cos^2 t + \sin^2 t = s^2 - 2p = s^2 + 2s - \frac{1}{2},$$

which leads to $s = -1 \pm \sqrt{10}/2$. Because $-2 < s < 2$, however, the only possible value for s is $-1 + \sqrt{10}/2$. Then

$$(1 - \sin t)(1 - \cos t) = 1 - s + p = \frac{5}{4} - 2s = \frac{13}{4} - \sqrt{10},$$

so $k + m + n = 27$.

8. (Answer: 085)

Suppose that $\frac{x+1}{y+1} = m$ holds for some integer $m > 1$, so that

$$x = my + (m - 1). \qquad (*)$$

Because y is a factor of x, it must also be a factor of $m - 1$. Hence there is a positive integer k with $m - 1 = ky$. Substitute $m = ky + 1$ into $(*)$ to find that

$$x = (ky + 1)y + ky = ky(y + 1) + y. \qquad (\dagger)$$

It follows from $x \le 100$ that

$$k \le \frac{100 - y}{y(y+1)}.$$

There are $\left\lfloor \frac{100 - y}{y(y+1)} \right\rfloor$ positive integers k that satisfy this inequality. For each positive integer y, equation $(\dagger)$ shows that there is a one-one correspondence between such k and ordered pairs (x, y) with the desired property. Hence the number of pairs is

$$\sum_{y=1}^{99} \left\lfloor \frac{100 - y}{y(y+1)} \right\rfloor = \sum_{y=1}^{9} \left\lfloor \frac{100 - y}{y(y+1)} \right\rfloor$$

$$= \left\lfloor \frac{99}{2} \right\rfloor + \left\lfloor \frac{98}{6} \right\rfloor + \left\lfloor \frac{97}{12} \right\rfloor + \left\lfloor \frac{96}{20} \right\rfloor + \left\lfloor \frac{95}{30} \right\rfloor + \left\lfloor \frac{94}{42} \right\rfloor + \left\lfloor \frac{93}{56} \right\rfloor + \left\lfloor \frac{92}{72} \right\rfloor + \left\lfloor \frac{91}{90} \right\rfloor$$
$$= 49 + 16 + 8 + 4 + 3 + 2 + 1 + 1 + 1$$

$$= 85.$$

9. (Answer: 616)
Let $\theta = \angle BAD$, so that $\angle BDM = 3\theta$ and $\angle ABD = 2\theta$. Combine the Law of Sines and the double-angle formula for the sine function to find that

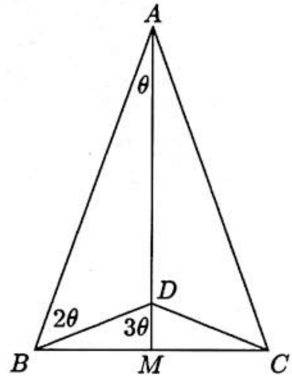
$$\frac{BD}{\sin\theta} = \frac{AD}{\sin 2\theta} = \frac{10}{2\sin\theta\cos\theta},$$

from which $\cos\theta = \dfrac{5}{BD}$ follows. Hence

$$AB = \frac{AM}{\cos\theta} = \frac{11}{5}BD.$$

Apply the Pythagorean Theorem to obtain

$$\left(\frac{11}{5}BD\right)^2 - 11^2 = BM^2 = BD^2 - 1^2.$$



It follows that $BD = \dfrac{5\sqrt{5}}{2}$, hence that $BM = \dfrac{11}{2}$ and $AB = \dfrac{11}{2}\sqrt{5}$. Thus the perimeter of $\triangle ABC$ is $2(AB + BM) = 11\sqrt{5} + 11 = \sqrt{605} + 11$, and $a + b = 616$.

Alternate Solution. Let the bisector of $\angle ABD$ intersect $\overline{AD}$ at E, and let $x = BE = AE$. By the Pythagorean Theorem,

$$BM = \sqrt{BE^2 - EM^2} = \sqrt{x^2 - (11-x)^2} = \sqrt{22x - 121}.$$

By applying the Pythagorean Theorem two more times, we find that

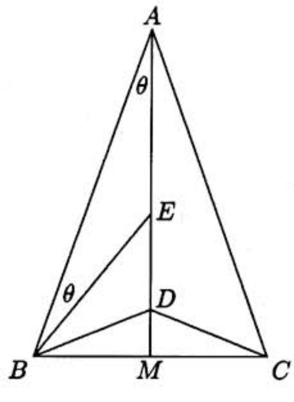
$$AB = \sqrt{BM^2 + AM^2} = \sqrt{22x} \quad \text{and}$$
$$BD = \sqrt{BM^2 + DM^2} = \sqrt{22x - 120}.$$

By the angle-bisector theorem, we have that

$$\frac{AB}{BD} = \frac{AE}{DE},$$

from which

$$\frac{\sqrt{22x}}{\sqrt{22x - 120}} = \frac{x}{10 - x}.$$



By squaring both sides of this equation and solving for x, we find that $x = 55/8$. Hence $BM = 11/2$ and $AB = (11/2)\sqrt{5}$. The perimeter of the triangle is $2(AB + BM) = 11\sqrt{5} + 11 = \sqrt{605} + 11$, so $a + b = 616$.

10. (Answer: 215)

Suppose that $N > 42$ is not the sum of a positive multiple of 42 and a positive composite integer. Let M be the smallest positive integer that makes $N - M$ divisible by 42. Because N is not divisible by 42, it follows that $M < 42$, and the conditions on N imply that no term of the arithmetic progression

$$M\ ,\ M+42\ ,\ M+84\ ,\ \ldots\ ,\ N-42$$

is composite. If there are at most four terms in this progression, then

$$N \leq (41 + 3 \cdot 42) + 42 = 209.$$

On the other hand, if there are more than four terms, then $M = 5$ is required, for there must be a multiple of 5 among any five consecutive terms of an arithmetic progression whose constant difference is not divisible by 5, and no term after M is composite. Thus the only progression with at least five terms begins

$$5\ ,\ 47\ ,\ 89\ ,\ 131\ ,\ 173\ ,\ 215\ ,\ \ldots\ ,$$

which has 215 as its first composite term. Thus 215 is the largest integer that is not the sum of a positive multiple of 42 and a composite positive integer.

Alternate Solution. Because 42 and 5 are relatively prime, every integer can be expressed in the form $42x+5y$, for some integers x and y. Moreover, (x, y) is a solution of

$$42x + 5y = n$$

if and only if $(x - 5, y + 42)$ is also a solution. Therefore, there is one solution for which $1 \leq x \leq 5$. It follows that the largest integer that cannot be written in the form $42x + 5y$ with $x \geq 1$ and $y \geq 2$ is $42 \cdot 5 + 5 \cdot 1 = 215$. In other words, every integer larger than 215 is the sum of a multiple of 42 and a composite number — a multiple of 5, in fact. Now check that $215 - 42 = 173$, $215 - 2 \cdot 42 = 131$, $215 - 3 \cdot 42 = 89$, $215 - 4 \cdot 42 = 47$, and $215 - 5 \cdot 42 = 5$ are prime, thereby showing that 215 is the largest integer that is not the sum of a positive multiple of 42 and a composite positive integer.

11. (Answer: 040)

Let x, y, z with $x \le y \le z$ be the sides of Q, the rectangular parallelepiped that is similar to P. Since Q is cut from P by a plane parallel to one of the faces of P, two of the numbers x, y, z must equal two of the numbers a, b, c. Furthermore

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c} < 1, \qquad (*)$$

so it follows that $y = a$ and $z = b$. Thus

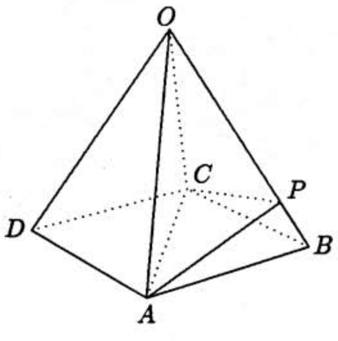
$$\frac{a}{b} = \frac{b}{c},$$

so $ac = b^2 = 1995^2 = 3^2 5^2 7^2 19^2$. Now 1995^2 has $(2+1)^4 = 81$ factors, and $a < c$ follows by substituting $y = a$ and $z = b$ into $(*)$. Thus the triple (a, b, c) can be selected in $(81-1)/2 = 40$ ways.

Each of these choices for a and c results in a rectangular parallelepiped of the type desired. Indeed, if $a < b = 1995 < c$ and $ac = 1995^2$, then cutting P by a plane parallel to the $a \times b$ face and at distance $x = a^2/b$ from that face produces an $x \times a \times b$ parallelepiped similar to P.

12. (Answer: 005)

Let P be the foot of the perpendicular from A to $\overline{OB}$. The pyramid's symmetry implies that P is also the foot of the perpendicular from C to $\overline{OB}$. Without loss of generality, we may assume $OP = 1$, from which $AP = PC = 1$, $OB = OA = \sqrt{2}$, and $BP = \sqrt{2} - 1$ follow. Two applications of the Pythagorean Theorem now give



$$AB^2 = AP^2 + BP^2 = 4 - 2\sqrt{2}$$

and

$$AC^2 = 2 \cdot AB^2 = 8 - 4\sqrt{2}.$$

The measure of the dihedral angle determined by faces OAB and OBC is the same as that of $\angle APC$. Use the Law of Cosines to obtain

$$\cos\angle APC = \frac{AP^2 + CP^2 - AC^2}{2 \cdot AP \cdot CP} = \frac{2 - 8 + 4\sqrt{2}}{2} = -3 + \sqrt{8}.$$

We then have $m + n = -3 + 8 = 5$.

Query. Without loss of generality, one may instead assume that $AB = 1$, so that $AC = \sqrt{2}$. Can you complete the calculation of $\cos\angle APC$?

13. (Answer: 400)

Let m be a positive integer. The largest integer n for which $f(n) = m$ is

$$\begin{aligned}\left\lfloor \left(m+\frac{1}{2}\right)^4 \right\rfloor &= \left\lfloor m^4 + 2m^3 + \frac{3}{2}m^2 + \frac{1}{2}m + \frac{1}{16} \right\rfloor \\ &= \left\lfloor m^4 + 2m^3 + \frac{1}{2}(3m^2+m) + \frac{1}{16} \right\rfloor \\ &= \left(m+\frac{1}{2}\right)^4 - \frac{1}{16},\end{aligned}$$

the last line following because $3m^2 + m$ is even. Therefore the number of integers n with $f(n) = m$ is

$$\left\lfloor \left(m+\frac{1}{2}\right)^4 \right\rfloor - \left\lfloor \left((m-1)+\frac{1}{2}\right)^4 \right\rfloor = \left(m+\frac{1}{2}\right)^4 - \left(m-\frac{1}{2}\right)^4 = 4m^3 + m.$$

Thus $f(n) = m$ for $4m^3 + m$ consecutive positive integers n. Now observe that $6^4 < 1995 < 7^4$, so that either $f(1995) = 6$ or $f(1995) = 7$. Because

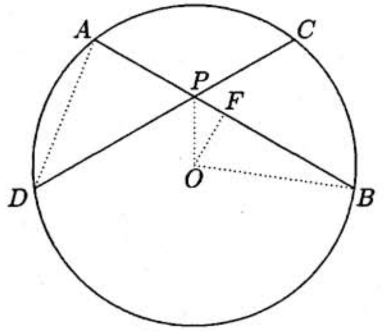
$$\sum_{m=1}^{6} (4m^3 + m) = 1785,$$

it follows that $f(1786) = f(1787) = \cdots = f(1995) = 7$, hence that

$$\begin{aligned}\sum_{k=1}^{1995} \frac{1}{f(k)} &= \sum_{k=1}^{1785} \frac{1}{f(k)} + \frac{1995-1785}{7} \\ &= \sum_{m=1}^{6} \frac{4m^3+m}{m} + 30 \\ &= \sum_{m=1}^{6} (4m^2+1) + 30 = 400.\end{aligned}$$

14. (Answer: 378)
Let the chords be denoted by $\overline{AB}$ and $\overline{CD}$, and the intersection of the chords by P, where $AP \leq BP$ and $CP \leq DP$. Let O be the center of the circle, and F be the foot of the perpendicular from O to $\overline{AB}$. By the Pythagorean Theorem,

$$\begin{aligned} OF &= \sqrt{OB^2 - FB^2} = \sqrt{42^2 - 39^2} \\ &= \sqrt{(42+39)(42-39)} = 9\sqrt{3}. \end{aligned}$$



Because $OP = 18$, it follows that

$$FP = 9 \text{ and } \angle OPB = 60°,$$

hence that

$$BP = BF + FP = 39 + 9 = 48$$

and $AP = 30$. A similar argument shows that $\angle OPD = 60°$, $DP = 48$, and $CP = 30$. Because $\triangle DPB$ is isosceles and $\angle DPB = 120°$, it follows that inscribed angle ABD is $30°$, and that central angle AOD is $60°$. Thus $\triangle AOD$ is equilateral, with $AD = 42$. The desired area is the sum of the area of $\triangle APD$ and the area of the segment of the circle bounded by $\overline{AD}$ and minor arc AD. This is

$$\begin{aligned} &\text{Area}(\triangle APD) + [\text{Area}(\text{Sector } AOD) - \text{Area}(\triangle AOD)] \\ =&\frac{1}{2}(30)(48)\sin 60° + \left[\frac{1}{6}\pi(42)^2 - \frac{1}{2}(42)^2 \sin 60°\right] \\ =&294\pi - 81\sqrt{3}. \end{aligned}$$

Therefore, $m + n + d = 294 + 81 + 3 = 378.$

15. (Answer: 037)

A *successful string* is a sequence of H's and T's in which $HHHHH$ appears before TT does. Each successful string must belong to one of the following three types:

(i) those that begin with T, followed by a successful string that begins with H;

(ii) those that begin with H, HH, HHH, or $HHHH$, followed by a successful string that begins with T;

(iii) the string $HHHHH$.

Let p_H denote the probability of obtaining a successful string that begins with H, and let p_T denote the probability of obtaining a successful string that begins with T. It follows that

$$p_T = \frac{1}{2}p_H \qquad \text{and} \qquad p_H = \left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}\right)p_T + \frac{1}{32}.$$

Solving these equations simultaneously, we find that

$$p_H = \frac{1}{17} \qquad \text{and} \qquad p_T = \frac{1}{34}.$$

Thus the probability of obtaining five heads before obtaining 2 tails is

$$p = p_H + p_T = \frac{3}{34}$$

and $m + n = 37$.